Dynamic Complementarity in Information Acquisition

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First Draft: Feb, 2015

April 20, 2016

Abstract

This paper studies an infinite-horizon environment in which overlapping generations of investors choose whether to acquire costly information about a stock's expected dividend. Multiple equilibria in information acquisition can arise in such an environment because current investors have more incentive to become informed if more investors are informed in the future, as the future stock payoff becomes more sensitive to the fundamental. This dynamic complementarity dominates classical static substitutability because the interim dividend payout makes the future stock payoff more sensitive to the fundamental compared with the current stock price. The multiplicity in information acquisition is more likely to arise if the stock fundamental is more persistent than the stock supply or if the public signal becomes less precise.

Keywords: Information acquisition; Financial markets; Dynamic complementarity; Multiplicity;
JEL classification Information acquisition; Financial markets; Dynamic complementarity; Multiplicity;

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1 Introduction

Investors in the financial market acquire private information and trade in order to beat the market. How do investors' information acquisition decisions interact with each other? In particular, is information a complement or a substitute? Grossman and Stiglitz (1980), in a static model, provide a classical view to this question: the fact that privately acquired information is partially revealed through prices means that the larger the share of informed investors today, the smaller the return to information acquisition. Thus, information is a *static substitute* in that its value decreases with the share of informed investors *today*. This substitutability in information acquisition leads to a unique equilibrium in Grossman and Stiglitz (1980). Following Grossman and Stiglitz (1980), the literature predominantly assumes that agents are allowed to acquire information only at the beginning of the economy. What if this assumption is relaxed and dynamic information acquisition is allowed? In particular, how do people's information choices interact *over time*? How does this dynamic aspect shapes people's incentive to acquire information?

This paper fills the gap by endogenizing information choice in an otherwise standard infinite-horizon asymmetric information trading model. In the model, there is a stock that pays a dividend each period. The dividend is stochastic and consists of a persistent component (the stock fundamental) and a transient component. The stock's supply also follows some persistent process. Members of each generation, upon their birth, freely observe the entire history of stock prices and dividends. They are then offered an opportunity to observe the history of stock fundamental at some cost.[1]

In this dynamic framework, information acquisition emerges naturally, not only as a *static substitute* but also as a *dynamic complement*. That is, agents' incentive to acquire information not only decreases with the share of informed agents *today* (which corresponds to the

[1]The physical and information structure is very close to Wang (1994), Watanabe (2008), and more recently, Albagli (2015), except that information choice is endogenous.

classical insight provided by Grossman and Stiglitz (1980)) but also increases with the share of informed agents *in the future*. This is because as more agents get informed in the future, future resale stock prices become more sensitive to the fundamental. This creates more uncertainty for today's uninformed agents (because they cannot perfectly observe today's fundamental) and raises the value of information today.

My main result is that this dynamic complementarity may dominate static substitutability and leads to multiplicity in information acquisition. The intuition is as follows. In this dynamic environment, multiple steady states arise if the conditional variance of the stock payoff, and thus the value of information, is increasing in the steady-state share of the informed investors. Now consider a marginal increase in the steady-state share of the informed. This has two opposing effects on the value of information. First, there are more informed investors today. Thus, the current stock price becomes a more precise signal about the fundamental. This tends to reduce the conditional variance of stock payoffs and thus the value of information. This is the classical static substitutability effect. The magnitude of this effect is proportional to the loading of the stock *price* on the fundamental.[2] Second, there are more informed investors in the future. This makes the future stock payoff more sensitive to the fundamental and thus increases the conditional variance of stock payoffs. This is the dynamic complementarity effect. The magnitude of this effect is roughly proportional to the loading of the stock *payoff* on the fundamental.[3] Since the stock payoff consists of both the *interim dividend payout* and the stock price and thus is *more* sensitive to the fundamental than just the stock price, the dynamic complementarity effect dominates the static substitutability effect, implying an upward-sloping value of information with respect to the steady-state share of informed investors, leading to multiplicity.

Second, I derive parameter restrictions under which multiplicity prevails. The finding here

[2]Roughly, this is because the variance of the price signal is proportional to the square of the loading coefficients on the fundamental of the current stock price.

[3]Similar to footnote 2, this is because the variance of the stock payoff is proportional to the square of the loading coefficients on the fundamental of the stock payoffs.

is that the fundamental needs to be sufficiently more persistent than supply for the multiplicity to arise. To understand this, note that in a dynamic framework, both the future fundamental and the future supply enter into the stock payoff, and marginal increases in today's share of informed investors may reduce the value of information by increasing the conditional correlation between the future fundamental and the future supply. The logic is as follows. Increasing the share of informed investors today makes the equilibrium price a more precise signal about the current stock fundamental but a noisier signal of the current stock supply.[4] This increases the conditional standard deviation of the stock supply faced by the uninformed. This in turn raises the conditional correlation between the future fundamental and the future supply. With a larger correlation, signals about the current stock fundamental becomes less useful in predicting future stock payoff because any signal that predicts a good future fundamental also predicts a large future stock supply. The former raises agents' estimate of the future stock payoff, whereas the latter lowers it. The two forces cancel out each other, thus leaving the signal less useful in predicting the future stock payoff. I show that the magnitude of this effect is tied to the persistence parameter of supply, and it is dominated by dynamic complementarity if and only if the fundamental is sufficiently more persistent than supply.

Last, I conduct various comparative statics exercises to examine how other parameters of the model affect the multiplicity result. In particular, I find that multiplicity is less likely to arise when the precision of the public signal improves. Thus, for a regulator that is aiming to stabilize asset markets, disclosing more precise public information is helpful because it eliminates equilibrium multiplicity. This result contributes to the recent debate on the desirability of the regulatory effort to provide more precise public information, such as the Sarbanes-Oxley Act and, more recently, the Dodd-Frank Act.

The paper is structured as follows. Section 2 sets up the model economy. Section 3 describes

[4] In Avdis (2014)'s model, this increases the value of information. Here, it tends to reduce the value of information.

the formal steps of the proof. Section 4 provides intuition for the main step of the proof. Section 5 conducts comparative statics excercises. Section 6 checks some robustness issues. Section 7 concludes.

Literature Review The paper is most closely related to the literature that studies *static* information acquisition choices in finite-horizon trading models. An early example is Froot et al. (1992). The broad scheme we share is that dynamic informational linkages lead to multiplicity. Relative to this literature, I make two contributions. The first is generality. I set up a fully standard dynamic environment as in Wang (1994) and Watanabe (2008) with endogenous information and characterize the subspace of parameters under which multiplicity in information acquisition prevails. In particular, I find that the persistence of the stock fundamental needs to be sufficiently high relative to the persistence of stochastic stock supply for the multiplicity to arise. This generalizes the result by Avdis (2014) that with random walk stochastic supply, multiplicity in information acquisition does not arise. I also clarify the role of the interim dividend payout in generating multiplicity.

Second, the literature assumes that information choice is made only at the very beginning of the world.[5] Thus, it is not clear whether the multiplicity result would still survive if dynamic information acquisition is allowed.[6] This paper shows, in an *infinite-horizon* framework, that multiplicity does arise with dynamic information acquisition. This is, however, due to the dynamic coordinations of information choices across generations, which is not present in a finite-horizon framework.

Recently, Avdis (2014) proposes an interesting mechanism that focuses on the information role of the stock supply: as the price signal becomes more informative about the stock fundamental, it necessarily becomes less informative about the stock supply. This creates more incentive to acquire information and leads to multiplicity. The source of multiplicity

[5] The literature proposes various interpretations for this assumption. For example, Froot et al. (1992) model it as explicit staggered execution, where orders are executed randomly across periods.

[6] In fact, in Cai (2015), I develope a repeated Grossman and Stiglitz (1980) economy with an arbitrary horizon and show that if the horizon is finite, the equilibrium is always unique.

is different here. To illustrate this, section 4.2 shows formally that 1) the information role of supply is also present in my formulation, and 2) does not play an essential role in my proof.

There is a literature that studies multiplicity in information acquisition using static models (Grossman and Stiglitz, 1980; Verrecchia, 1982; Veldkamp, 2006a,b; Chamley, 2007; Barlevy and Veronesi, 2007; Ganguli and Yang, 2009; Cespa and Vives, 2014). Grossman and Stiglitz (1980) and Verrecchia (1982) obtain the classical result of strategic substitution in information acquisition. Later works identify various sources of strategic complementarity in information acquisition. Barlevy and Veronesi (2007) argue that with correlated fundamentals and noise trading complementarity may arise. Ganguli and Yang (2009) illustrate that complementarity may result when agents own private information about their endowment. Veldkamp (2006a,b) generates complementary by embedding an increasing returns to scale information production sector into an otherwise standard noisy rational expectations model. All of the previous models are static.

Last, the theory is also related to the literature that studies exogenous asymmetric information trading models in an infinite horizon, pioneered by Wang (1993, 1994) and Campbell and Kyle (1993). It is particularly related to models that study overlapping generations of investors (Spiegel, 1998; Bacchetta and Van Wincoop, 2006; Watanabe, 2008; Biais et al., 2010; Albagli, 2015)[7]. Although the physical structure of my paper is very close to these papers, in my model the information acquisition choice is endogenous. Dow and Gorton (1994) study a dynamic overlapping-generations model with private information where, similar to this paper, a dynamic informational linkage is present: information gets incorporated into the price only if informed traders expect future traders to also impound their information in the price. Unlike this paper, however, it does not concern the issue of multiplicity.[8]

[7]This literature identifies high volatility equilibria and low volatility equilibria with different stock price sensitivity with respect to noise trader risks.

[8]I think an anonymous referee for pointing this out.

2 Model Economy

Time is discrete and runs from $-\infty$ to $+\infty$. The economy is populated by a continuum of overlapping generations risk-averse agents who consume a single consumption good. The good is treated as the numeraire. There are two assets in the economy: a bond in perfect elastic supply, paying a return R;[9] and a stock that pays a dividend

$$D_t = F_t + \varepsilon_t^D \tag{2.1}$$

each period. F_t is the persistent component of the dividend process. Later I call F_t the stock fundamental. The stock fundamental follows an AR(1) process:

$$F_t = \rho^F F_{t-1} + \varepsilon_t^F, 0 \le \rho^F \le 1. \tag{2.2}$$

The stock supply, x_t, follows an AR(1) process as well:

$$x_t = \rho^x x_{t-1} + \varepsilon_t^x, 0 \le \rho^x \le 1. \tag{2.3}$$

As in Wang (1994), I assume that there is a public signal every period about the current fundamental:

$$S_t = F_t + \varepsilon_t^S. \tag{2.4}$$

The shock vector $\boldsymbol{\varepsilon_t} = [\varepsilon_t^D, \varepsilon_t^F, \varepsilon_t^x, \varepsilon_t^S]$ is i.i.d. over time, with mean 0 and covariance matrix $diag(\sigma_D^2, \sigma_F^2, \sigma_x^2, \sigma_S^2)$.

Investors live for two periods. When they are born, they are endowed with a certain amount of wealth and also observe the entire history of the dividend and stock price. They are then offered an opportunity to acquire information at some cost χ. If they choose to acquire

[9] Alternatively one can interpret the bond as a storage technology without nonnegative constraint.

information, they also observe the history of the stock fundamental. I call investors who choose to acquire information the "informed" investors and the rest "uninformed." The information set of the generation-t uninformed is

$$\Omega_t^U = \{P_s, D_s, S_s\}_{s=-\infty}^t,$$

and that for the informed is

$$\Omega_t^I = \{P_s, D_s, S_s, F_s\}_{s=-\infty}^t.$$

As is standard in this class of models, an informed investor, observing the history of the fundamental and stock price, can perfectly deduce the stock supply. For uninformed investors, their conditional expectations are derived from Kalman filter equations. We use $\hat{F}$ and $\hat{x}$ to denote the conditional mean of the current fundamental and stock supply for the uninformed:

$$\hat{F}_t = E(F_t|\Omega_t^U) \tag{2.5}$$

$$\hat{x}_t = E(x_t|\Omega_t^U). \tag{2.6}$$

After the information acquisition stage, the financial market opens and trade occurs. After that, old investors exit and consume their wealth. The timeline is summarized in figure 1.

The individual agents' problem is as follows. First, they make their information acquisition choice:

$$V = \max\{V^I, V^U\},$$

where V^I denotes the expected utility of generation-t informed investors, and V^U denotes the expected utility for the generation-t uninformed. V^I and V^U are in turn determined by

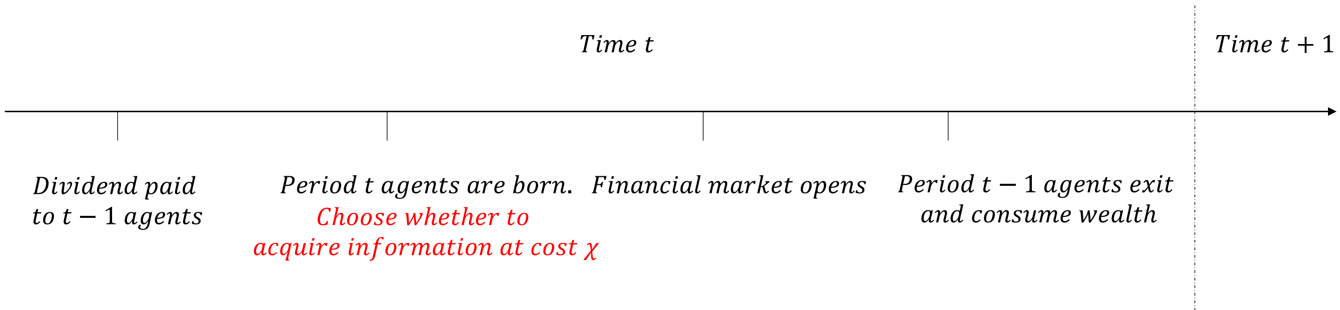


Figure 1: Timeline

agents' portfolio choice:

$$\begin{aligned} V^i &= \int_P V^i(P) dH_1(P) \\ V^i(P) &= \max_{e,b,c'} \int_{P',F',\boldsymbol{\varepsilon}'} U(c') dH_2(P', F', \boldsymbol{\varepsilon'} | \Omega^i) \\ eP + b &\leq w - \mathbb{1}\{i = I\}\chi \\ c' &\leq (D(F', \boldsymbol{\varepsilon'}) + P')e + Rb, \end{aligned}$$

where $U(c) = -\exp(-\alpha c)$, α is the risk-averse parameter. $D(F, \boldsymbol{\varepsilon}) = F + \varepsilon^D$, H_1, H_2 are determined in general equilibrium.

2.1 Equilibrium Definition

As is standard in the literature, I will focus on the stationary equilibrium.

Definition 2.1 *Denote* $s = \{\hat{F}, F, x\}$. *A steady state is* $\{P(s), \lambda, \{e_i(s), b_i(s)\}_{i=U,I}\}$ *s.t.*

1. *$e_i(s), b_i(s)$ solves the uninformed and informed agents' problem given $P(s)$.*

2. *The market clears: $\lambda e_I(s) + (1-\lambda)e_U(s) = x(s), \forall s_t, t$.*

3. *$V_U = V_I$ if $\lambda \in (0,1)$; if $\lambda = 0$, $V_U \geq V_I$; if $\lambda = 1$, $V_U \leq V_I$,*

where $\hat{F}$ is the conditional expectation given by equation 2.5. The last condition guarantees that agents' information choice is optimal. For instance, if there is a positive fraction of both informed and uninformed investors ($\lambda \in (0,1)$), it has to be the case that the expected utility of the informed and the expected utility of the uninformed are equalized.

It is challenging to solve a noisy rational expectations Model with general, potentially nonlinear, price functions. Hence, in later analysis, I accord with the literature and restrict my attention to the class of linear equilibrium. I conjecture that the price function takes the following form:

Definition 2.2 *A linear equilibrium is a steady state where price functions are linear with respect to their arguments. That is, there exists $\{a, p_{\hat{F}}, p_F, p_x\}$ such that*

$$P(s) = a + p_{\hat{F}}\hat{F} + p_F F - p_x x. \tag{2.7}$$

Except as otherwise noted, in later sections I will restrict attention to linear equilibrium.

3 Multiplicity in Information Acquisition

The purpose of this section is to formally establish that there is multiplicity in information acquisition in this economy (theorem 1). To do so, I take the following steps:

1. First define steady states where information (i.e. the fraction of informed investors λ) is exogenous. Denote it as the exogenous-information steady state $\Phi(\lambda)$ (definition 3.1).

2. At each $\Phi(\lambda)$ compute the difference in the expected utility of the informed and the expected utility of the uninformed. Denote it as the value of information $\pi(\lambda)$ (definition 3.2).

3. If $\pi(\lambda)$ is equal to some measure of the utility cost of acquiring information (unless at boundary), $\Phi(\lambda)$ is a steady state (lemma 3.2).

4. If there are multiple values of λ, we find multiple steady states (lemma 3.3 and theorem 1).

Step 1: Define the exogenous-information steady state $\Phi(\lambda)$

Intuitively, the auxiliary concept of the exogenous-information steady state is just a steady state where all generations of investors' information is *exogenously* fixed, as studied in Spiegel (1998), Watanabe (2008), and Biais et al. (2010):

Definition 3.1 *An exogenous-information steady state given λ is $\Phi(\lambda) = \{P(s), \lambda, \{s_i(s), b_i(s\}_{i=U,I}\}$ such that it satisfies condition 1 and 2 in definition 2.1.*

As is well known in this literature, there exists multiple exogenous-information linear steady states given any λ. To make the exposition transparent, I will focus on the *high-volatility* equilibrium in the main part of the analysis[10]. Note that I do not take a stand on which equilibria one *should* select, as both low-volatility and high-volatility equilibria have desirable properties. The purpose of focusing on the high-volatility equilibrium is to illustrate that there is a new source of multiplicity associated with agents' information choice.

One may wonder about the existence of the exogenous-information steady state. As in Spiegel (1998), I provide conditions such that there exists a unique exogenous-information steady state at least locally near $\lambda = 0$.

Assumption 1

$$(R-\rho^x)^2 - 4\alpha^2\sigma_x^2\left(\left(1+\frac{\rho^F}{R-\rho^F}(\theta_2+\theta_3)\right)^2\left[\left(\rho^F\right)^2\Sigma_0+\sigma_F^2\right]+\left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2+\left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2\right)>0,$$

where Σ_0 is given by A.26, θ_2 is given by A.28, and θ_3 is given by A.29.

Lemma 3.1 *Under assumption 1, a high-volatility exogenous-information steady state $\Phi(\lambda)$ exists and is unique for λ sufficiently close to 0.*

[10] The insight obtained in the main part of the paper carries over to low-volatility equilibria as well, as shown in the robustness check section.

Proof. See the appendix. ■

In later analysis, I assume that assumption 1 holds.

Step 2: Define the value of information $\pi(\lambda)$

The value of information is the ratio of the expected utilities of the informed and uninformed *net of information cost* at each high-volatility exogenous-information steady state indexed by λ.

Definition 3.2 *Denote the expected utility of the informed* W^I *and uninformed* W^U *at each* $\Phi(\lambda)$*. Define the following*

$$\pi(\lambda) = W^U/W^I,$$

where $W^i, i = I, U$ *are given by*

$$\begin{aligned}
W^i &= \textstyle\int_P W^i(P) dF_t(P) \\
W^i(P) &= \max_{e,b,c'} \textstyle\int_{P',F',\boldsymbol{\varepsilon}'} U(c') dH(P', F', \boldsymbol{\varepsilon}'|\Omega^i) \\
eP + b &\leq w \\
c' &\leq (D(F', \boldsymbol{\varepsilon}') + P')e + Rb.
\end{aligned}$$

Step 3: Compare the value of information $\pi(\lambda)$ **with some measure of the information cost** The next lemma shows that the value of information function $\pi(\lambda)$ allows us to directly compare the expected gain from acquiring information with the cost of acquiring information and determine whether $\phi(\lambda)$ is a steady state.

Lemma 3.2 $\forall\lambda \in (0, 1)$*,* $\Phi(\lambda)$ *is a steady state if and only if*

$$\pi(\lambda) = \exp(\alpha R \chi).$$

For $\lambda = 0$ (1)*,* $\Phi(\lambda)$ *is a steady state if and only if*

$$\pi(\lambda) \leq (\geq) \exp(\alpha R \chi).$$

Proof. Choose the case $\lambda \in (0,1)$. Verifying the other cases is similar. It can be shown that under Constant absolute risk aversion utility, $V^U = W^U$; $V^I = W^I e^{\alpha R \chi}$. Thus, if $\pi(\lambda) = e^{\alpha R \chi}$ holds, then

$$\frac{V^U}{V^I} = \frac{W^U}{W^U e^{\alpha R \chi}} = \frac{\pi(\lambda)}{e^{\alpha R \chi}} = 1.$$

Thus all of the conditions for a steady state hold for $\Phi(\lambda)$, and $\Phi(\lambda)$ is a steady state. ■

Step 4: Prove multiplicity

The goal of the last step is to show that (under some conditions) there exists multiple values of λ satisfying the conditions stated in lemma 3.2.

To do so, let me state the following lemma:

Lemma 3.3 *Suppose*

$$(1-\theta_0)\rho^F > \rho^x + \phi \tag{3.1}$$

for some $\theta_0 \in [0,1]$ given by A.30 and ϕ given by A.44, then

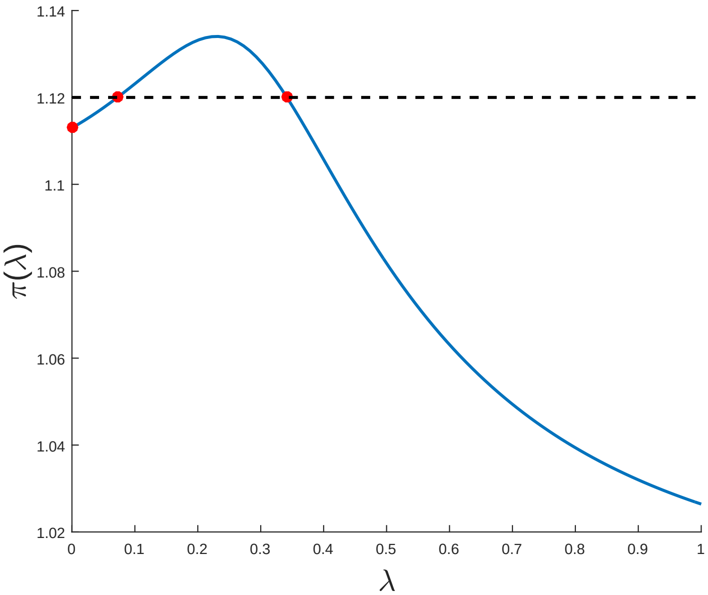
$$\frac{d\pi(\lambda)}{d\lambda} > 0 \text{ for } \lambda \text{ sufficiently small.}$$

$\theta_0, \rho^F, \rho^x, \phi$ are either structural parameters or functions of the structural parameters. the lemma says that under certain conditions (which we will elaborate in the next section), the incentive of people to become informed, $\pi(\lambda)$, increases as there are more informed investors. This is in sharp contrast with the classical substitution effect in Grossman and Stiglitz (1980).

The next theorem establishes the multiplicity result:

Theorem 1 *Under condition 3.1, there exits χ such that multiple steady states exist.*

Proof. The proof follows from the intermediate value theorem and lemma 3.3. The value of information function $\pi(\lambda)$ is differentiable, and hence continuous. Given that $\pi'(\lambda) > 0$ for λ sufficiently small, choose χ such that $e^{\alpha \chi} = \pi(\lambda_1)$ for some λ_1 sufficiently small but strictly positive. Then we know that λ_1 is a steady state. Also, we know that $\pi(0) < \pi(\lambda_1) = e^{\alpha \chi}$. Thus, we know



The blue curve depicts numerically solved $\pi(\lambda)$. The black dashed line depicts the information cost. Red dots are numerically solved steady states. Parameter values: $\alpha = 1, R = 1.05, \rho^F = 1, \rho^x = 0, \sigma_F^2 = 0.01, \sigma_D^2 = 0.1, \sigma_x^2 = 0.01$.

Figure 2: The value of information $\pi(\lambda)$

that $\lambda_0 = 0$ is another steady state because no one is informed and the gain from acquiring information is less than the cost. Thus there are at least two steady states (see figure 2). ■

4 Intuition and a Heuristic Proof of Lemma 3.3

The key step toward multiplicity is lemma 3.3. A rigorous proof is delegated to the appendix. This section is devoted to explaining the main steps and intuition for why the value of information is locally increasing ($\pi'(\lambda) > 0$) for sufficiently small λ.

To begin, as in Grossman and Stiglitz (1980), the value of information is given by the ratio of the stock payoff uncertainty faced by the uninformed and informed. More precisely, the value of information is given by

$$\pi = \sqrt{\frac{Var(D' + P'|\Omega^U)}{Var(D' + P'|\Omega^I)}},$$

where D' is the next period dividend and P' is the next period stock price.

In this heuristic proof, instead of focusing on the ratio of uncertainty, we proxy the value of information with the *difference* in the conditional variance of the stock payoffs between the uninformed and informed:

$$\Delta V := Var(D' + P'|\Omega^U) - Var(D' + P'|\Omega^I).$$

To derive an expression for ΔV, plug equation 2.1 and equation 2.7 into the stock payoff $D' + P'$:

$$\begin{aligned} D' + P' &= \underbrace{F' + \varepsilon'^D}_{D'} + \underbrace{a + p_{\hat{F}}\hat{F}' + p_F F' - p_x x'}_{P'} \\ &= a + (1 + p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon'^D. \end{aligned} \tag{4.1}$$

Then plug in the law of motion of F' (equation 2.2), the law of motion of x' (equation 2.3), and the law of motion of $\hat{F}'$ (equation A.5). Then rearranging, one obtains

$$\begin{aligned} D' + P' &= a + (1 + p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon'^D \\ &= \underbrace{a + e_1\hat{F} + e_2\hat{x}}_{\text{common knowledge}} + \underbrace{e_3F - e_4x}_{\text{known to the informed}} + \underbrace{e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime} + e_8\varepsilon^{S\prime}}_{\text{shock}}, \end{aligned} \tag{4.2}$$

where coefficients e_i are given in A.6 through A.13 and are all positive. $\hat{F} = E(F|\Omega^U)$ and $\hat{x} = E(x|\Omega^U)$ are the uninformed's estimate of the current period fundamental and the stock supply respectively. As shown in expression 4.2, we can decompose the equation into three components. The first component, $a + e_1\hat{F} + e_2\hat{x}$, is common knowledge. The third component, $e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime} + e_8\varepsilon^{S\prime}$, consists of future shocks that no one could possibly know today. The second component, however, is only known to the informed. Thus, the difference in uncertainty between informed and uninformed is just the conditional volatility of the second component:

$$\begin{aligned} \Delta V &= Var(D' + P'|\Omega^U) - Var(D' + P'|\Omega^I) \\ &= Var(e_3F - e_4x|\Omega^U) \\ &= e_3^2Var(F|\Omega^U) + e_4^2Var(x|\Omega^U) - 2e_3e_4Cov(F, x|\Omega^U). \end{aligned} \tag{4.3}$$

The first two variance terms reflect, respectively, the predictive role of the fundamental and supply. The last term is negative, reflecting the fact that conditional on observing the current stock price, fundamental and supply are correlated and the bigger the correlation, the lower the value of information. Taking derivatives with respect to each term, we have

$$\frac{\partial \Delta V}{\partial \lambda} = \frac{\partial e_3^2 Var(F|\Omega^U)}{\partial \lambda} + \frac{\partial e_4^2 Var(x|\Omega^U)}{\partial \lambda} - \frac{\partial 2 e_3 e_4 Cov(F, x|\Omega^U)}{\partial \lambda}. \tag{4.4}$$

Thus, whether ΔV is locally increasing in λ depends on three terms. Next I will examine the value of each term, taking λ very close to zero.

4.1 The predictive role of fundamental F

The first term $\partial e_3^2 Var(F|\Omega^U)/\partial \lambda$ in equation 4.4 reflects how perturbations in λ affect the value of information through the predictive role of fundamental F. When the share of informed, λ, increases, two opposing forces affect the value of $e_3^2 Var(F|\Omega^U)$. On the one hand, there is classic substitutability: that there are more informed investors today implies a more informative current stock price. Thus, the conditional variance of fundamental $Var(F|\Omega^U)$ is reduced. On the other hand, since there are more informed investors in the future, the future stock price loads more heavily on the fundamental, and thus the loading coefficient e_3 increases.

To compare the two forces, the crucial observation is that classical substitutability is absent when $\lambda \to 0$. More precisely:

$$\lim_{\lambda \to 0} \frac{\partial Var(F|\Omega^U)}{\partial \lambda} = 0. \tag{4.5}$$

This is due to the nature of the Kalman filter: $Var(F|\Omega^U)$ depends on p_F and p_x only to the extent that it depends on $(\frac{p_F}{p_x})^2$. This implies that the derivative of $Var(F|\Omega^U)$ with respect to λ is proportional to $\frac{p_F}{p_x}$. Also note that when $\lambda \to 0$, $p_F \to 0$ because there are no informed investors that knows perfectly the value of F. Therefore,

$$\frac{\partial Var(F|\Omega^U)}{\partial \lambda} \propto \frac{p_F}{p_x} \to 0 \text{ when } \lambda \to 0.$$

Thus, perturbing λ near $\lambda = 0$ does not affect the conditional uncertainty about the current fundamental faced by the uninformed.

On the other hand, the loading coefficient e_3^2 is *strictly* increasing in λ. To see this, we need to derive an expression for e_3. For simplicity, ignore the law of motion for $\hat{F}'$ and just plug in the law of motion for F' and x' into equation 4.2:

$$\begin{aligned} D' + P' &= a + (1+p_F)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon'^D \\ &= a + (1+p_F)(\rho^F F + \varepsilon^{F\prime}) - p_x(\rho^x x + \varepsilon^{x\prime}) + p_{\hat{F}}\hat{F}' \\ &= a + (1+p_F)\rho^F F - p_x \rho^x x + p_{\hat{F}}\hat{F}' + (1+p_F)\varepsilon^{F\prime} - p_x \varepsilon^{x\prime}. \end{aligned} \tag{4.6}$$

Thus, e_3 and e_4 are approximately

$$e_3 \approx \rho^F(1+p_F) \tag{4.7}$$

$$e_4 \approx \rho^x p_x. \tag{4.8}$$

As $\lambda \to 0$, the loading of stock price on the fundamental, p_F, converges to 0, but the loading of the stock *payoffs* on the fundamental, e_3, converges to some strictly positive number ρ^F. This is because of the presence of the interim dividend payout, which introduces additional sensitivity into the future stock payoff compared with the current stock price (see the second term in equation 4.1). Thus, the derivative of e_3^2 with respect to λ is proportional to $2e_3$, which in turn is roughly equal to $2\rho^F(1+p_F)$ which converges to some strictly positive number when λ is sufficiently small:

$$\frac{\partial e_3^2}{\partial \lambda} > 0 \text{ when } \lambda \to 0. \tag{4.9}$$

Combining the static substitutability (equation 4.5) and the dynamic complementarity (equation 4.9), one can show that the first term in equation 4.4 is always positive at the limit:

$$\lim_{\lambda\to 0} \frac{\partial[e_3^2 Var(F|\Omega^U)]}{\partial \lambda} = \underbrace{\lim_{\lambda\to 0} \underbrace{\frac{\partial e_3^2}{\partial \lambda}}_{>0} \underbrace{Var(F|\Omega^U)}_{>0}}_{\text{dynamic complementarity}>0} + \underbrace{\lim_{\lambda\to 0} \underbrace{\frac{\partial Var(F|\Omega^U)}{\partial \lambda}}_{=0} e_3^2}_{\text{static substitutability}=0} > 0$$

4.2 The predictive role of supply x

The second term $\partial e_4^2 Var(x|\Omega^U)/\partial\lambda$ in equation 4.4 captures the predictive role of supply x. As discussed in Avdis (2014), the equilibrium price becomes a noisier signal of supply when there are more informed investors. This force tends to increase the conditional supply uncertainty faced by the agents and thus increase the value of information. More precisely, one can show that

$$Var(x|\Omega^U) = (\frac{p_F}{p_x})^2 Var(F|\Omega^U).$$

As λ increases, the stock price becomes more sensitive to the fundamental, and thus the ratio $\frac{p_F}{p_x}$ increases. This tends to push up the conditional uncertainty of supply and thus increases the value of acquiring information and leads to multiplicity.

In this model, however, this effect is absent locally around $\lambda = 0$. Again, this result follows from the nature of the Kalman filter: like $Var(F|\Omega^U)$, $Var(x|\Omega^U)$ is also an (implicit) function of $(\frac{p_F}{p_x})^2$. Thus the derivative of $Var(x|\Omega^U)$ with respect to λ is proportional to $\frac{p_F}{p_x}$. As a result, it tends toward zero as λ tends toward zero:

$$\frac{\partial Var(x|\Omega^U)}{\partial\lambda} \propto \frac{p_F}{p_x} \to 0, \text{as } \lambda \to 0.$$

In fact, one can further strengthen the statement by observing that when $\lambda \to 0$, the stock price becomes a perfect signal of supply. This implies that there is no uncertainty about the stock supply locally around $\lambda = 0$. That is $\lim_{\lambda\to 0} Var(x|\Omega^U) = 0$. To sum up

$$\lim_{\lambda\to 0} \frac{\partial[e_4^2 Var(x|\Omega^U)]}{\partial\lambda} = \underbrace{\lim_{\lambda\to 0} \frac{\partial e_4^2}{\partial\lambda} \underbrace{Var(x|\Omega^U)}_{=0}}_{=0} + \underbrace{\lim_{\lambda\to 0} \underbrace{\frac{\partial Var(x|\Omega^U)}{\partial\lambda}}_{=0} e_4^2}_{=0} = 0 \tag{4.10}$$

Thus, in this model, the supply channel does not play any role locally around $\lambda = 0$. This is different from the work by Avdis (2014) that emphasizes the supply channel. We can safely ignore this term henceforth.

4.3 Correlation between the fundamental and supply as an offsetting force

The third term $-\partial e_3 e_4 Cov(F, x|\Omega^U)/\partial\lambda$ in equation 4.4 reflects the fact that an increase in λ may reduce the value of information through the correlation channel. The logic is as follows. When λ increases from zero to some strictly positive increment, the price becomes a noisier signal of stock supply. This increases the standard deviation of the stock supply relative to that of the stock fundamental and thus the conditional correlation between the fundamental and supply. When the correlation increases, information about the fundamental is not that valuable because any signal that predicts a good fundamental also predicts a large stock supply. The two forces cancel out each other, making the signal less useful in predicting the future stock payoff.

To derive the sign of $\frac{\partial \Delta V}{\partial \lambda}$, we need to compare the third term with the first term (recall that the second term vanishes as λ tends to zero). The rough intuition is the following. As can be seen in equation 4.7, the loading of the future stock payoffs on the current fundamental, e_3, is proportional to ρ^F. Thus, e_3^2 is proportional to $(\rho^F)^2$. When taking derivatives in equation 4.4, we can factor out the constants and thus the first term, $\partial e_3^2 Var(F|\Omega^U)/\partial\lambda$, is proportional to $(\rho^F)^2$. Similarly, the third term,$-\partial e_3 e_4 Cov(F, x|\Omega^U)/\partial\lambda$, is proportional to $\rho^F \rho^x$. Therefore, roughly speaking, for the first term to dominate the third term we need $(\rho^F)^2$ to be greater than $\rho^F \rho^x$, or equivalently, ρ^F greater than ρ^x.

When the exact formulas of e_3 and e_4 are plugged in, one can formally show that when $\lambda \to 0$,

$$\frac{\partial \Delta V}{\partial \lambda} > 0 \iff (1-\theta_0)\rho^F > \rho^x, \tag{4.11}$$

where the expression of θ_0 is given in A.30 and is always between 0 and 1. Thus, for ΔV to be locally increasing in λ, it is necessary that the fundamental is sufficiently more persistent than supply.

The $1-\theta_0$ term measures the *information advantage* of the informed at $\lambda = 0$[11]. It is always

[11] Formally, it captures how sensitive informed agents' estimate of the fundamental is with respect to the true fundamental relative to the uninformed's:

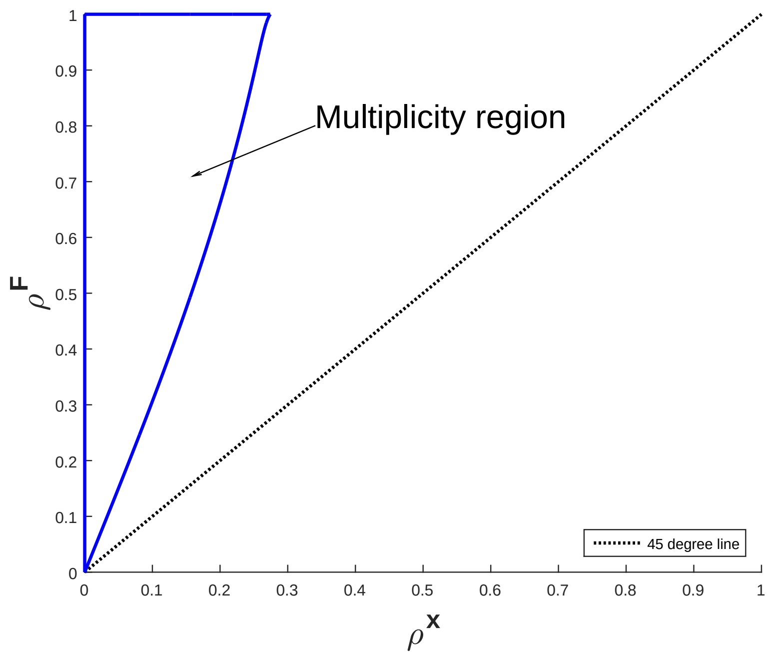
$$1-\theta_0 = \lim_{\lambda\to 0} \frac{\partial [E(F|\Omega^I) - E(F|\Omega^U)]}{\partial F}.$$

nonnegative because informed investors are always at a (weak) information advantage relative to the uninformed. The presence of $1-\theta_0$ in condition 4.11 says that multiplicity is more likely to arise when the information advantage of the informed is larger. The intuition is as follows. Whether multiplicity arises depends on the strength of the dynamic complementarity, or how much more sensitive the future stock price would become with respect to the fundamental upon a marginal increase in λ. When the information advantage of the informed is very large, the informed's demand is much more sensitive to the fundamental than the uninformed. Thus, a marginal increase in the share of the informed tomorrow makes tomorrow's aggregate demand, and hence tomorrow's stock price, much more sensitive to the fundamental. This implies that the dynamic complementarity is stronger, and thus multiplicity is more likely to arise.

So far we have used ΔV as a proxy for the value of information and illustrate a necessary and sufficient condition for multiplicity to arise (condition 4.11). In the rigorous proof, there is an additional term ϕ in condition 3.1. This ϕ term captures the level effect of changing λ. Namely, perturbing λ changes not only the *difference* in uncertainty but also the *level* of uncertainty. This has a nontrivial effect on the value of information. To see this,

$$\begin{aligned}\pi &= \sqrt{\frac{Var(D'+P'|\Omega^U)}{Var(D'+P'|\Omega^I)}} \\ &= \sqrt{1+\frac{Var(D'+P'|\Omega^U)-Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^I)}} \\ &= \sqrt{1+\frac{\Delta V}{Var(D'+P'|\Omega^I)}}.\end{aligned} \tag{4.12}$$

Thus, the value of information is a monotonic function of $\frac{\Delta V}{Var(D'+P'|\Omega^I)}$. Not only the difference but also the level of uncertainty $Var(D'+P'|\Omega^I)$ enter into the expression of π and matter. The term ϕ is complicated and hard to characterize analytically. Yet, as shown in figure 3, where I numerically solve and plot the parameter pair (ρ^F, ρ^x) that satisfies condition 3.1, the general insight that the fundamental should be more persistent carries over.



This figure plots the parameter subspace (ρ^x, ρ^F) *where condition 3.1 is satisfied (the area surrounded by the blue curve), and hence multiplicity in information acquisition could arise.*
Parameter values: $R = 1.05, \alpha = 1, \sigma_F^2 = \sigma_D^2 = \sigma_x^2 = \sigma_S^2 = 0.01$

Figure 3: Multiplicity Region

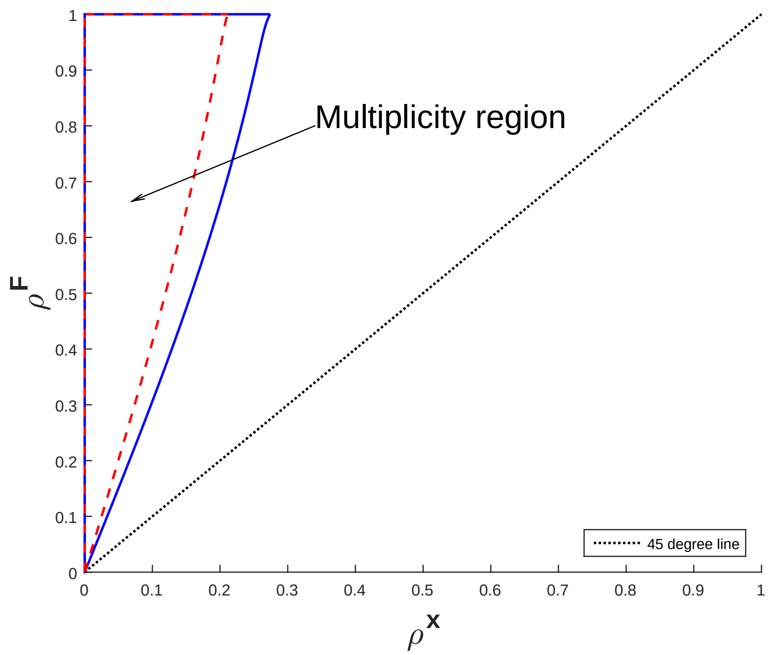
5 Comparative Statics with an Application to Disclosure

So far we have illustrated that for the multiplicity in information acquisition to arise, it is crucial that the fundamental is more persistent than supply. This section investigates how other parameters, $(\sigma_F^2, \sigma_D^2, \sigma_x^2, R, \alpha)$ and most crucially σ_S^2, affect the multiplicity result.

As shown in figure 4, increasing the precision of the public signal shrinks the multiplicity region. This result can be understood as follows. As discussed before, $1 - \theta_0$ captures the information advantage of the informed. When the public signal becomes more precise, this information advantage vanishes. This makes condition 3.1 harder to satisfy. Thus, the multiplicity region shrinks.

This result provides an interesting perspective on recent policy attempts to provide more precise public information. It says, for a regulator seeking to stabilize asset prices, it is desirable to disclose more precise public information because it helps to eliminate equilibrium multiplicity.

Other comparative statics exercises are collected in figure 5. All the results can be understood by

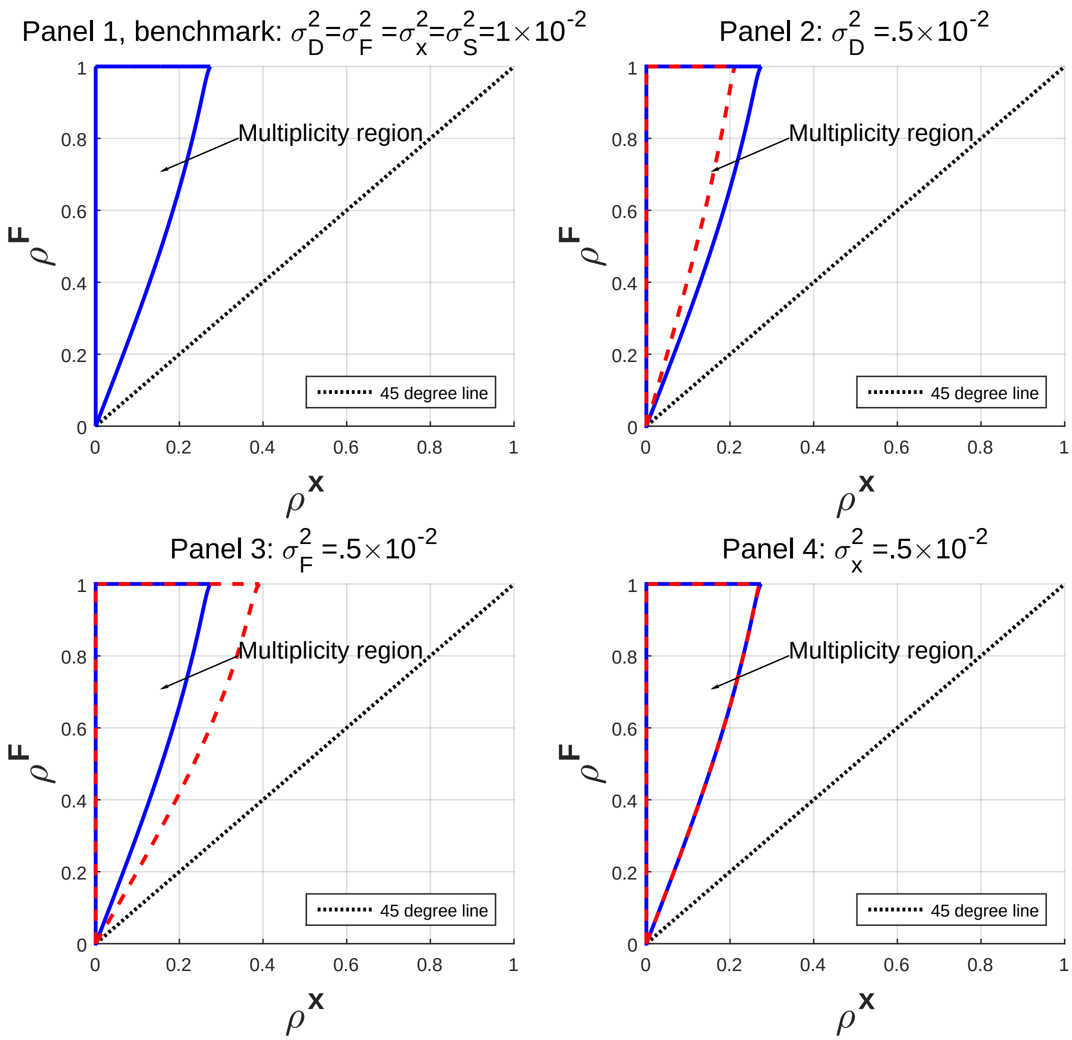


This figure plots how the multiplicity region changes when σ_s^2 decreases from 0.01 to 0.005. The blue area plots the benchmark case whereas the red dashed area plots the case where $\sigma_s^2 = 0.005$.

Figure 4: Comparative Statics: $\sigma_s^2 = 0.01 \rightarrow 0.005$

examining how these parameters affect the value of θ_0 and hence condition 4.11. First, reducing the dividend noise σ_D^2 shrinks the multiplicity region, since the dividend becomes a more precise signal of the fundamental and therefore θ_0 increases. Likewise, a decrease of σ_F^2 reduces prior uncertainty and therefore makes uninformed agents rely less upon dividend information. This reduces the sensitivity parameter θ_0. Thus, the multiplicity region expands. Last, varying σ_x^2 hardly affects the multiplicity region because the price signal contains pure noise when there are no informed agents. Therefore, price signals drop out of uninformed agents' Bayesian updating problem for the fundamental (although it is still useful in predicting future supply). Thus, varying σ_x^2 does not change the value of θ_0. Thus, the multiplicity region hardly changes.[12] Similarly, risk averse coefficient α and interest rate parameter R do not affect multiplicity region, because they do not enter into the expression of θ_0.

[12] Varying σ_x^2 does affect the multiplicity region through the feasibility condition (equation 1). This effect is present only when the volatility parameters are set large. This case is checked in the robustness check section.



Panels 2-4 plot how the multiplicity region changes when σ_D^2, σ_F^2, and σ_x^2 change from 0.01 *to* 0.005*, respectively. The blue area is the benchmark case. The red areas in panels 2,3 and 4 are the new multiplicity regions when the volatility parameters are set to* 0.005*.*

Figure 5: Comparative Statics

6 Robustness

Large volatility parameters

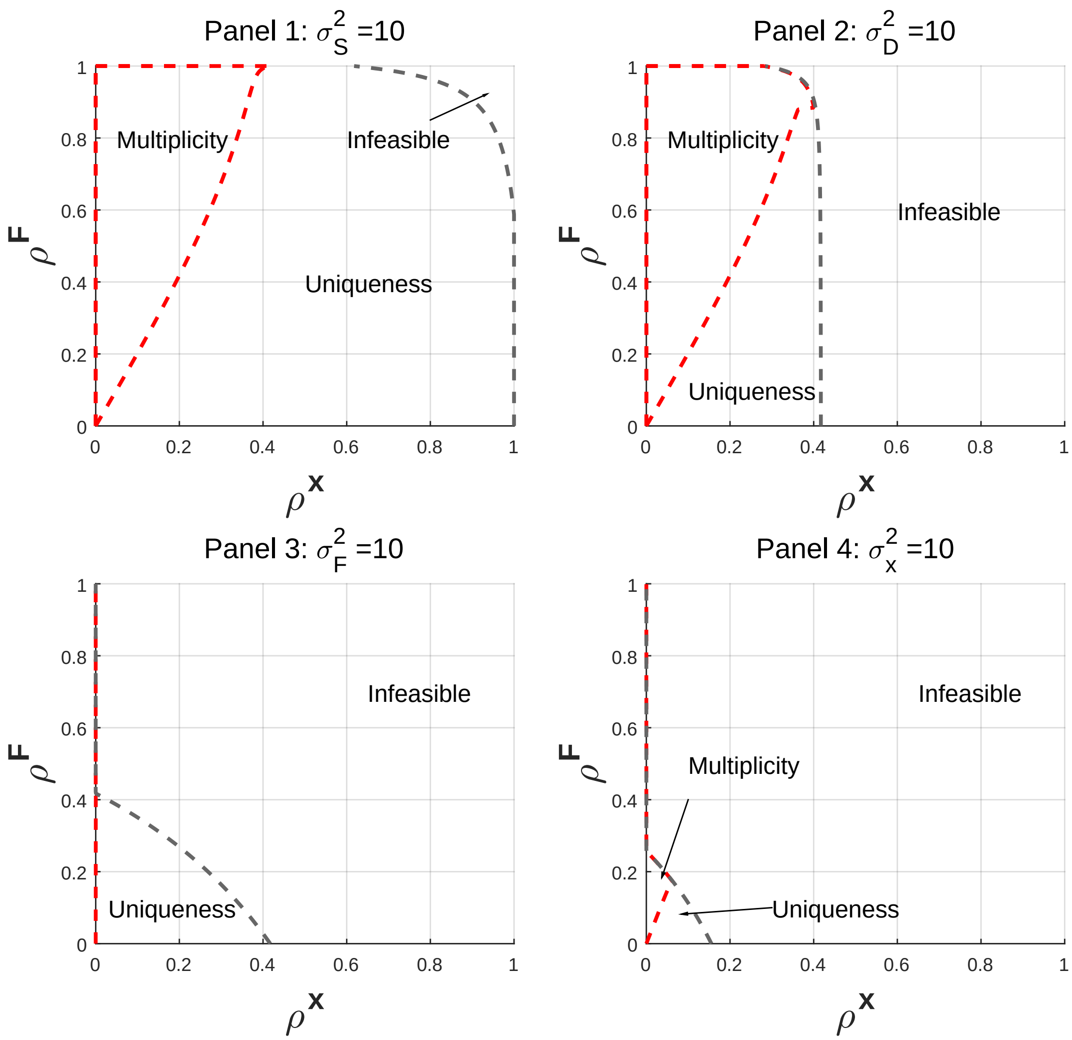
In the main part of the paper, the volatility parameters are set to be small ($\sigma_F^2 = \sigma_D^2 = \sigma_x^2 = \sigma_S^2 = 0.01$). Small volatility is convenient because the feasibility constraint (assumption 1) never binds for any feasible $(\rho^F, \rho^x) \in [0,1] \times [0,1]$. This section shows what the multiplicity region looks like when the volatility parameters are set to be large.

When the volatility parameters are set to be large, the feasibility constraint (assumption 1) kicks in, which affects the multiplicity region. However, the general conclusion that the fundamental needs to be more persistent than supply for multiplicity to arise is robust. To illustrate, I set volatility parameters to 10 one by one and plot the resulting new multiplicity region (figure 6). The multiplicity region is surrounded by a red dashed line. The feasibility constraint becomes most stringent when σ_F^2 and σ_x^2 are set to be large. This is in line with the literature studying overlapping-generations asymmetric information trading models, which typically set σ_x^2 to be a very small number.

Low Volatility Equilibria

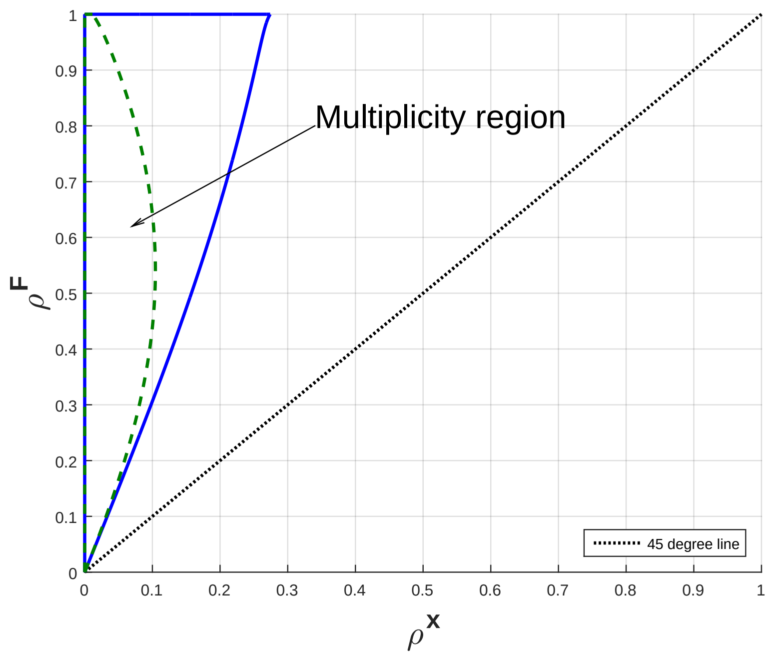
In the main part of the paper, I focus on high-volatility equilibria to illustrate the multiplicity in information acquisition. This is convenient because the ϕ term becomes very small with high-volatility equilibria.[13] This section checks whether the main results of the paper extend to the low-volatility equilibria. The answer is yes. To illustrate, I set the parameters to the benchmark level and plot the multiplicity region corresponding to high- and low-volatility equilibria, respectively. As can be seen in figure 7, the multiplicity region (surrounded by dashed green lines) shrinks with low-volatility equilibria. Note that the multiplicity region displays backward bending with low-volatility equilibrium, due to the *level* effect captured by the ϕ term in condition 3.1. Specifically, it is possible that an increase in the share of the informed induces an increase in the loading of the

[13] To understand this, note that ϕ is proportional to $1/V^U$. In the high-volatility equilibrium, $V^U = Var(P' + D'|\Omega^U)$ is generally very big. Thus, $\phi \approx 0$.



Panels 2-4 plots how the multiplicity region changes when σ_D^2, σ_F^2, and σ_x^2 change from 0.001 *to* 10 *respectively. The blue area in panel 1 is the benchmark case where $\sigma_F^2 = \sigma_D^2 = \sigma_x^2 = 0.001$. The red area is the new multiplicity region after, say, σ_F^2 is changed to* 50*. In panels 2 through 4, all areas to the right of the grey dashed curves violate assumption 1 and are thus infeasible.*

Figure 6: Robustness Check I: large volatility parameters



The figure plots the multiplicity region under the benchmark parameter values $\sigma_F^2 = \sigma_D^2 = \sigma_x^2 = 0.001$. The blue area depicts the multiplicity region with high-volatility equilibria. whereas the dashed green area depicts the multiplicity region with low-volatility equilibria.

Figure 7: Robustness Check II: Low-volatility equilibria

equilibrium price on supply p_x, which leads to higher uncertainty and thus, by equation 4.12, lower value of information. When the persistence parameter ρ^F gets bigger, this level effect becomes more powerful due to a stronger dynamic linkage.

Last, it is natural to ask how robust the multiplicity result is when agents live for more than two periods. This question, however, cannot be addressed within the current framework because when agents live for more than two periods, their incentives to acquire information depends on the conditional mean of stock fundamental. The conditional mean, however, is affected by random shocks (see equation A.5). This in turn implies that agents' incentives to acquire information become functions of random shocks, as do the price coefficients. With random price coefficients, the equilibrium price is no longer normally distributed. This breaks down the classic linear-normal framework and calls for an alternative approach.[14]

[14] Avdis (2014) studies information acquisition with long-lived agents. There information market only opens at the beginning of the economy. Crucially, the prior mean is normalized to be zero. Thus, the model remains tractable.

7 Conclusion

This paper studies implications of dynamic information acquisition in an otherwise standard infinite-horizon asymmetric information trading model. It is shown that multiplicity in information acquisition could arise in such an environment. This finding is due to the dynamic complementarity in information acquisition: current investors have more incentive to become informed if more investors are informed in the future. The dynamic complementarity dominates classical static substitutability (Grossman and Stiglitz (1980)) and leads to multiplicity. The interim dividend payout is important because it introduces additional sensitivity with respect to the fundamental into the future stock payoff compared with the current stock price. It is also crucial that the fundamental is more persistent than the stock supply.[15] The model has some other implications. For example, multiplicity in information acquisition becomes less likely to arise when the public signal becomes more precise. This suggests that for a regulator seeking to stabilize asset prices, disclosing more precise public information is beneficial because it helps to eliminate equilibrium multiplicity.

[15] Wang (1994) and Watanabe (2008) set the persistence parameters to be equal. Albagli (2015) and Biais et al. (2010) choose the fundamental to be more persistent than supply.

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A Appendix: Proofs

Proof of lemma 3.1

I prove existence by construction. Namely, I find a system of equations that fully characterizes exogenous-information steady states $\Phi(\lambda)$ given any value of λ. The method I use to look for such a system of equation is very similar to Wang (1994): look for a system of equations of $\left(\Sigma, p_{\hat{F}}, p_F, p_x\right)$ given λ. Σ is the uninformed's prior of the state variable (F, x) and is pinned down by the Kalman filter equations, given $p_{\hat{F}}, p_F, p_x$. $p_{\hat{F}}, p_F, p_x$ are pinned down by the market clearing condition, given Σ. The detailed proof is as follows.

First, we would like to look for an equation characterizing Σ. The state s_t evolves and the signal is given by the following dynamic system:

$$
\begin{aligned}
s_{t+1} &= As_t + w_{t+1}, \text{ where } w_{t+1} \tilde{} N(0, Q) \\
y_{t+1} &= Gs_{t+1} + v_{t+1}, \text{ where } v_{t+1} \tilde{} N(0, R),
\end{aligned}
$$

where

$$s_t = \begin{bmatrix} F_t \\ x_t \end{bmatrix}$$

$$y_t = \begin{bmatrix} S^P_{t+1} \\ D_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} p_F F_{t+1} - p_x x_{t+1} \\ F_{t+1} + \varepsilon^D_{t+1} \\ F_{t+1} + \varepsilon^S_{t+1} \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} \rho^F & 0 \\ 0 & \rho^x \end{bmatrix} \\ G &= \begin{bmatrix} p_F & -p_x \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \\ Q &= \begin{bmatrix} \sigma_F^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix} \\ R &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_D^2 & 0 \\ 0 & 0 & \sigma_S^2 \end{bmatrix}. \end{aligned}$$

Apply the formula of the Kalman filter:

$$\begin{aligned} \hat{s}_{t+1} &= E\left(s_{t+1}|y_{t+1}, y_t, ...\right) = A\hat{s}_t + \left[A\Sigma_t A'G' + QG'\right]\left[GA\Sigma_t A'G' + GQG' + R\right]^{-1}\left(y_{t+1} - GA\hat{s}_t\right) \qquad (A.1) \\ \Sigma_{t+1} &= Var\left(s_{t+1}|y_{t+1}, y_t, ...\right) = A\Sigma_t A' + Q - \left[A\Sigma_t A'G' + QG'\right]\left[GA\Sigma_t A'G' + GQG' + R\right]^{-1}\left[A\Sigma_t A'G' + QG'\right]'. \end{aligned}$$

At a stationary equilibrium, Σ must be stationary over time, and therefore the second equation becomes

$$\Sigma = A\Sigma A' + Q - \left[A\Sigma A'G' + QG'\right]\left[GA\Sigma A'G' + GQG' + R\right]^{-1}\left[A\Sigma A'G' + QG'\right]'. \qquad (A.2)$$

We arrive at the first equation, which characterizes the uninformed's conditional expectation Σ.

Next, we know that investors solve the following problem:

$$\max_e -E\left[e^{-\alpha Rw - \alpha e\left(D'+P'-RP\right)}|\Omega^i\right], i = u, i.$$

Thus, its demand is given by

$$\begin{aligned} D^i &= \frac{E\left[D' + P'|\Omega^i\right] - RP}{\alpha Var\left[D' + P' - RP|\Omega^i\right]} \qquad (A.3) \\ &= \frac{E\left[D' + P'|\Omega^i\right] - RP}{\alpha Var\left[D' + P'|\Omega^i\right]}, \end{aligned}$$

where the second equality follows because the current price P is in the information set of both agents. Note that $E\left[D' + P'|\Omega^i\right]$ and $Var\left[D' + P' - RP|\Omega^i\right]$ are the conditional mean and variance of the excess stock return perceived by the agent conditional on its information set Ω^i. We can write out $D' + P'$:

$$\begin{aligned} D' + P' &= F' + \varepsilon^{D\prime} + a + p_{\hat{F}}\hat{F}' + p_F F' - p_x x' \qquad (A.4) \\ &= a + \left(1 + p_F\right)F' + p_{\hat{F}}\hat{F}' - p_x x' + \varepsilon^{D\prime}. \end{aligned}$$

Note that by the previous Kalman filter expression,

$$\begin{aligned}\hat{F}' &= \rho^F\hat{F} + \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}\begin{bmatrix} p_F\left(F'-\rho^F\hat{F}\right) - p_x\left(x'-\rho^x\hat{x}\right) \\ F'-\rho^F\hat{F}+\varepsilon^{D\prime} \\ F'-\rho^F\hat{F}+\varepsilon^{S\prime}\end{bmatrix} \\ &= \rho^F\hat{F} + Y_1\left[p_F\left(F'-\rho^F\hat{F}\right) - p_x\left(x'-\rho^x\hat{x}\right)\right] + Y_2\left[F'-\rho^F\hat{F}+\varepsilon^{D\prime}\right] + Y_3\left[F'-\rho^F\hat{F}+\varepsilon^{S\prime}\right] \\ &= \left(\rho^F - Y_1p_F\rho^F - (Y_2+Y_3)\rho^F\right)\hat{F} + (Y_1p_F+Y_2+Y_3)F' - Y_1p_xx' + Y_1p_x\rho^x\hat{x} + Y_2\varepsilon^{D\prime} + Y_3\varepsilon^{S\prime},\end{aligned} \tag{A.5}$$

where $\begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}$ denotes the first column of matrix $[A\Sigma A'G'+QG'][GA\Sigma A'G'+GQG'+R]^{-1}$. Plug A.5 back into A.4, one obtains:

$$\begin{aligned}D'+P' &= a + (1+p_F)F' + p_{\hat{F}}\left[\left(\rho^F - Y_1p_F\rho^F - (Y_2+Y_3)\rho^F\right)\hat{F} + (Y_1p_F+Y_2+Y_3)F' - Y_1p_xx' + Y_1p_x\rho^x\hat{x} + Y_2\varepsilon^{D\prime} + Y_3\varepsilon^{S\prime}\right] \\ &\quad -p_xx' + \varepsilon^{D\prime} \\ &= a + p_{\hat{F}}\left(\rho^F - Y_1p_F\rho^F - (Y_2+Y_3)\rho^F\right)\hat{F} + p_{\hat{F}}Y_1p_x\rho^x\hat{x} + \left(1+p_F+p_{\hat{F}}(Y_1p_F+Y_2+Y_3)\right)F' \\ &\quad -\left(p_{\hat{F}}Y_1p_x+p_x\right)x' + \left(p_{\hat{F}}Y_2+1\right)\varepsilon^{D\prime} + p_{\hat{F}}Y_3\varepsilon^{S\prime} \\ &= a + p_{\hat{F}}\left(\rho^F - Y_1p_F\rho^F - (Y_2+Y_3)\rho^F\right)\hat{F} + p_{\hat{F}}Y_1p_x\rho^x\hat{x} + \rho^F\left(1+p_F+p_{\hat{F}}(Y_1p_F+Y_2+Y_3)\right)F - \rho^x\left(p_{\hat{F}}Y_1p_x+p_x\right)x \\ &\quad + \left(1+p_F+p_{\hat{F}}(Y_1p_F+Y_2+Y_3)\right)\varepsilon^{F\prime} - \left(p_{\hat{F}}Y_1p_x+p_x\right)\varepsilon^{x\prime} + \left(p_{\hat{F}}Y_2+1\right)\varepsilon^{D\prime} + p_{\hat{F}}Y_3\varepsilon^{S\prime}.\end{aligned}$$

To ease exposition, define

$$e_1 = p_{\hat{F}}\left(\rho^F - Y_1p_F\rho^F - (Y_2+Y_3)\rho^F\right) = p_{\hat{F}}\rho^F\left(1 - Y_1p_F - Y_2 - Y_3\right) \tag{A.6}$$

$$e_2 = p_{\hat{F}}Y_1p_x\rho^x \tag{A.7}$$

$$e_3 = \rho^F\left(1+p_F+p_{\hat{F}}(Y_1p_F+Y_2+Y_3)\right) \tag{A.8}$$

$$e_4 = \rho^x\left(p_{\hat{F}}Y_1p_x+p_x\right) \tag{A.9}$$

$$e_5 = 1+p_F+p_{\hat{F}}(Y_1p_F+Y_2+Y_3) \tag{A.10}$$

$$e_6 = p_{\hat{F}}Y_1p_x+p_x \tag{A.11}$$

$$e_7 = p_{\hat{F}}Y_2+1 \tag{A.12}$$

$$e_8 = p_{\hat{F}}Y_3 \tag{A.13}$$

(A.14)

Thus,

$$D'+P' = a + e_1\hat{F} + e_2\hat{x} + e_3F - e_4x + e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime}. \tag{A.15}$$

We can further simplify this expression by substituting out $\hat{x}$:

$$\begin{aligned}p_FF - p_xx &= p_F\hat{F} - p_x\hat{x} \\ \hat{x} &= \frac{p_F}{p_x}\hat{F} - \frac{p_F}{p_x}F + x.\end{aligned}$$

Thus,

$$\begin{aligned}D'+P' &= a + e_1\hat{F} + e_2\left(\frac{p_F}{p_x}\hat{F} - \frac{p_F}{p_x}F + x\right) + e_3F - e_4x + e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime} + e_8\varepsilon^{S\prime} \\ &= a + \left(e_1 + e_2\frac{p_F}{p_x}\right)\hat{F} + \left(e_3 - e_2\frac{p_F}{p_x}\right)F + (-e_4+e_2)x + e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime} + e_8\varepsilon^{S\prime}.\end{aligned}$$

Given this expression, the conditional expectation for the informed is given by

$$E\left(D'+P'|\Omega^I\right) = a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)F+\left(-e_4+e_2\right)\hat{x} \tag{A.16}$$

$$Var\left(D'+P'|\Omega^I\right) = e_5^2\sigma_F^2+e_6^2\sigma_x^2+e_7^2\sigma_D^2+e_8^2\sigma_S^2 \tag{A.17}$$

The conditional expectation for the uninformed is given by

$$E\left(D'+P'|\Omega^U\right) = a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)\hat{F}+\left(-e_4+e_2\right)\hat{x} \tag{A.18}$$

$$Var\left(D'+P'|\Omega^U\right) = H\Sigma H'+e_5^2\sigma_F^2+e_6^2\sigma_x^2+e_7^2\sigma_D^2+e_8^2\sigma_S^2, \tag{A.19}$$

where $H=\left[\begin{array}{cc} e_3 & -e_4\end{array}\right]=\left[\begin{array}{cc}\rho^F\left(1+p_F+p_{\hat{F}}\left(Y_1p_F+Y_2+Y_3\right)\right) & -\rho^x\left(p_{\hat{F}}Y_1p_x+p_x\right)\end{array}\right]$; Σ solves equation A.2.

Now the market clearning is given by

$$\lambda D^I+\left(1-\lambda\right)D^U=x.$$

Plug in the demand function (A.3):

$$\lambda\frac{E\left[D'+P'|\Omega^I\right]-RP}{\alpha Var\left[D'+P'|\Omega^I\right]}+\left(1-\lambda\right)\frac{E\left[D'+P'|\Omega^I\right]-RP}{\alpha Var\left[D'+P'|\Omega^I\right]}=x.$$

Plug in the conditional expectation and variance from equations A.16 and A.18:

$$\lambda\frac{a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)F+\left(-e_4+e_2\right)x-RP}{\alpha V^I}+\left(1-\lambda\right)\frac{a+\left(e_1+e_2\frac{p_F}{p_x}\right)\hat{F}+\left(e_3-e_2\frac{p_F}{p_x}\right)\hat{F}+\left(-e_4+e_2\right)x-RP}{\alpha V^U}=x$$

where $V^I=Var\left(D'+P'|\Omega^I\right); V^U=Var\left(D'+P'|\Omega^U\right).$

Rearranging and matching coefficients, we have three equations determining $p_{\hat{F}},p_F,p_x$:

$$\lambda\frac{e_1+e_2\frac{p_F}{p_x}}{\alpha V^I}+\left(1-\lambda\right)\frac{e_1+e_3}{\alpha V^U}=\left(\lambda\frac{R}{\alpha V^I}+\left(1-\lambda\right)\frac{R}{\alpha V^U}\right)p_{\hat{F}} \tag{A.20}$$

$$\lambda\frac{e_3-e_2\frac{p_F}{p_x}}{\alpha V^I}=\left(\lambda\frac{R}{\alpha V^I}+\left(1-\lambda\right)\frac{R}{\alpha V^U}\right)p_F \tag{A.21}$$

$$\lambda\frac{-e_4+e_2}{\alpha V^I}+\left(1-\lambda\right)\frac{-e_4+e_2}{\alpha V^U}-1=-\left(\lambda\frac{R}{\alpha V^I}+\left(1-\lambda\right)\frac{R}{\alpha V^U}\right)p_x \tag{A.22}$$

Note that we only need to keep track of one of $p_{\hat{F}}$ and p_F. To see this, sum equations A.20 and A.21:

$$\begin{aligned} e_1+e_3 &= \left(p_{\hat{F}}+p_F\right)R \\ p_{\hat{F}}\rho^F\left(1-Y_1p_F-Y_2\right)+\rho^F\left(1+p_F+p_{\hat{F}}\left(Y_1p_F+Y_2\right)\right) &= \left(p_{\hat{F}}+p_F\right)R \\ p_{\hat{F}}\rho^F+\rho^F+\rho^Fp_F &= \left(p_{\hat{F}}+p_F\right)R \end{aligned}$$

$$p_{\hat{F}}+p_F=\frac{\rho^F}{R-\rho^F}. \tag{A.23}$$

Thus, we only need to know, say, p_F, and we can deduce $p_{\hat{F}}=\frac{\rho^F}{R-\rho^F}-p_F$. Hence, for any λ, the exogenous information steady state $\Phi(\lambda)$ is (Σ, p_F, p_x) characterized by equations A.2, A.21, and A.22.

Next, I show that when $\lambda = 0$, the system of equations A.2, A.21, and A.22 can be explicitly solved. First, note that by equation A.21, $p_F = 0$. Second, by A.23, $p_{\hat{F}} = \frac{\rho^F}{R-\rho^F}$. To find p_x, rearrange A.22:

$$\begin{aligned}
\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda) \frac{1}{\alpha V^U}\right](-e_4 + e_2 + R p_x) &= 1 \\
\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda) \frac{1}{\alpha V^U}\right]\left(-\rho^x \left(p_{\hat{F}} Y_1 p_x + p_x\right) + p_{\hat{F}} Y_1 p_x \rho^x + R p_x\right) &= 1 \\
\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda) \frac{1}{\alpha V^U}\right](-\rho^x p_x + R p_x) &= 1
\end{aligned}$$

$$(R - \rho^x)\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda) \frac{1}{\alpha V^U}\right] p_x = 1. \tag{A.24}$$

Next, we need to look for expressions of V^I, V^U. To do so, we need to first find the matrix

$$\Sigma = \begin{bmatrix} Var\left(F|\Omega^U\right) & Cov\left(F, x|\Omega^U\right) \\ Cov\left(F, x|\Omega^U\right) & Var\left(x|\Omega^U\right) \end{bmatrix}.$$

Note that everyone, including the uninformed, observes the price signal $S^P = p_F F - p_x x$. Thus

$$Cov\left(F, x|\Omega^U\right) = Cov\left(F, \frac{p_F F - S^P}{p_x}|\Omega^U\right) = \frac{p_F}{p_x} Var\left(F|\Omega^U\right)$$

$$Var\left(x|\Omega^U\right) = Var\left(\frac{p_F F - S^P}{p_x}|\Omega^U\right) = \left(\frac{p_F}{p_x}\right)^2 Var\left(F|\Omega^U\right).$$

Thus, we only need to find $Var\left(F|\Omega^U\right)$ to determine Σ. Denote $Var\left(F|\Omega^U\right) = \Sigma_F$.

Thus,

$$\Sigma = \begin{bmatrix} 1 & \frac{p_F}{p_x} \\ \frac{p_F}{p_x} & \left(\frac{p_F}{p_x}\right)^2 \end{bmatrix} \Sigma_F$$

Then,

$$A \Sigma A' = \begin{bmatrix} \left(\rho^F\right)^2 & \rho^F \rho^x \frac{p_F}{p_x} \\ \rho^F \rho^x \frac{p_F}{p_x} & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \end{bmatrix} \Sigma_F$$

$$\begin{aligned}
A \Sigma A' + Q &= \begin{bmatrix} \left(\rho^F\right)^2 & \rho^F \rho^x \frac{p_F}{p_x} \\ \rho^F \rho^x \frac{p_F}{p_x} & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \end{bmatrix} \Sigma_F + \begin{bmatrix} \sigma_F^2 & \\ & \sigma_x^2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \\ \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2 \end{bmatrix}
\end{aligned}$$

$$A\Sigma A'G' + QG' = \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \\ \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2 \end{bmatrix} G'$$

$$= \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \\ \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2 \end{bmatrix} \begin{bmatrix} p_F & 1 & 1 \\ -p_x & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_F \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right] - p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ p_F \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] - p_x \left[(\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2\right] & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \end{bmatrix}$$

Now:

$$GA\Sigma A'G' + GQG' + R = \begin{bmatrix} p_F & -p_x \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \{p_F \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right] -p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right]\} & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ \{p_F \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] -p_x \left[(\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2\right]\} & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \end{bmatrix} + \begin{bmatrix} 0 & & \\ & \sigma_D^2 & \\ & & \sigma_S^2 \end{bmatrix}$$

$$= \begin{bmatrix} (p_F)^2 \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right] - 2 p_F p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] + (p_x)^2 \left[(\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2\right] & p_F \left[\left(\rho^F\right)^2 a_{11} + \sigma_F^2\right] -p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] & p_F \left[\left(\rho^F\right)^2 a_{11} + \sigma_F^2\right] -p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] \\ p_F \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right] - p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ p_F \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right] - p_x \left[\rho^F \rho^x \frac{p_F}{p_x} \Sigma_F\right] & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix}$$

Note that

$$GA\Sigma A'G' + GQG' + R = \begin{bmatrix} (p_x)^2 \sigma_x^2 & 0 & 0 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix} \text{ when } p_F = 0$$

Now we can write out equation A.2 explicitly, which, when $\lambda = 0$ and thus $p_F = 0$, is reduced to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Sigma_F = \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F \\ \rho^F \rho^x \frac{p_F}{p_x} \Sigma_F & (\rho^x)^2 \left(\frac{p_F}{p_x}\right)^2 \Sigma_F + \sigma_x^2 \end{bmatrix} -$$

$$\begin{bmatrix} 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ -p_x \sigma_x^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} (p_x)^2 \sigma_x^2 & 0 & 0 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 0 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 \\ 0 & 0 & 0 \end{bmatrix}'$$

The determinant when $\lambda = 0$ is

$$\Theta = \left|\left[GA\Sigma A'G' + GQG' + R\right]\right| = (p_x)^2 \sigma_x^2 \left[\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)^2\right] \tag{A.25}$$

Thus,

$$
\begin{aligned}
\left[GA\Sigma A'G' + GQG' + R\right]^{-1} &= \begin{bmatrix} (p_x)^2\sigma_x^2 & 0 & 0 \\ 0 & \left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_D^2 & \left(\rho^F\right)^2\Sigma_F + \sigma_F^2 \\ 0 & \left(\rho^F\right)^2\Sigma_F + \sigma_F^2 & \left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_S^2 \end{bmatrix}^{-1} \\
&= \frac{1}{\Theta}\begin{bmatrix} \frac{\Theta}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & (p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_S^2\right) & (p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right) \\ 0 & (p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right) & (p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_D^2\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_D^2\right)}{\Theta} \end{bmatrix} \text{ when } \lambda = 0
\end{aligned}
$$

Plug in these terms and pick the first entry of the matrices. One obtains that Σ_F must solve the following equation when $\lambda = 0$:

$$
\Sigma_F = \left(\rho^F\right)^2\Sigma_F + \sigma_F^2 - \frac{\left(\sigma_S^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right)^2}{\left[\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right)^2\right]} \tag{A.26}
$$

Denote the solution to this equation Σ_0 (we rule out negative roots). Intuitively, Σ_0 is the conditional mean of F for the uninformed when there are no informed investors.

We still need to obtain some expression for Y_1,Y_2, and Y_3 when $\lambda = 0$. Remember that they are entries of the first column of matrix $\left[A\Sigma A'G' + QG'\right]\left[GA\Sigma A'G' + GQG' + R\right]^{-1}$.

When $\lambda = 0$:

$$
\left[A\Sigma A'G' + QG'\right]\left[GA\Sigma A'G' + GQG' + R\right]^{-1} = \begin{bmatrix} 0 & \left[\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2\right]\left(\frac{(p_x)^2\sigma_x^2\sigma_S^2}{\Theta}\right) & \left[\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2\right]\left(\frac{(p_x)^2\sigma_x^2\sigma_D^2}{\Theta}\right) \\ \cdots & \cdots & \cdots \end{bmatrix}
$$

Thus, we can take out each element of the first row of the matrix:

$$
\theta_1 = Y_{1|\lambda=0} = 0 \tag{A.27}
$$

$$
\theta_2 = Y_{2|\lambda=0} = \frac{\sigma_S^2\left[\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2\right]}{\left[\left(\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2\right)^2\right]} \tag{A.28}
$$

$$
\theta_3 = Y_{3|\lambda=0} = \frac{\sigma_D^2\left[\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2\right]}{\left[\left(\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2\Sigma_0 + \sigma_F^2\right)^2\right]} \tag{A.29}
$$

Denote

$$
\theta_0 = \theta_2 + \theta_3 \tag{A.30}
$$

.

Crucially, note that

$$\begin{aligned}
\theta_0 &= \theta_2 + \theta_3 \\
&= \frac{\left(\sigma_D^2 + \sigma_S^2\right)\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right]}{\left[\left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 + \sigma_D^2\right)\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right)^2\right]} \\
&= \frac{\left(\sigma_D^2 + \sigma_S^2\right)\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right]}{\left[\left(\sigma_D^2 + \sigma_S^2\right)\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right] + \sigma_D^2\sigma_S^2\right]} \in [0,1]
\end{aligned}$$

Now we are ready to derive expressions for V^I and V^U :

$$\begin{aligned}
V^I &= e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2 + e_8^2\sigma_S^2 \\
&= \left(1 + p_F + p_{\hat{F}}\left(Y_1 p_F + Y_2 + Y_3\right)\right)^2 \sigma_F^2 + \left(-\left(p_{\hat{F}} Y_1 p_x + p_x\right)\right)^2 \sigma_x^2 + \left(p_{\hat{F}} Y_2 + 1\right)^2 \sigma_D^2 + \left(p_{\hat{F}} Y_3\right)^2 \sigma_S^2 \\
&\to \left(1 + p_{\hat{F}}\left(Y_2 + Y_3\right)\right)^2 \sigma_F^2 + p_x^2\sigma_x^2 + \left(p_{\hat{F}} Y_2 + 1\right)^2 \sigma_D^2 + \left(p_{\hat{F}} Y_3\right)^2 \sigma_S^2 \\
&= \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \sigma_F^2 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2 + p_x^2\sigma_x^2
\end{aligned}$$

$$\begin{aligned}
V^U &= H\Sigma H' + e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2 + e_8^2\sigma_S^2 \\
&\to \left[\begin{array}{cc} \rho^F\left(1 + p_{\hat{F}}\left(Y_2 + Y_3\right)\right) & -\rho^x p_x \end{array}\right]\left[\begin{array}{cc} \Sigma_0 & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} \rho^F\left(1 + p_{\hat{F}}\left(Y_2 + Y_3\right)\right) & -\rho^x p_x \end{array}\right]' \\
&\quad + \left(1 + \frac{\rho^F}{R - \rho^F}\frac{\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2}{\sigma_D^2 + \left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2}\right)^2 \left[\sigma_F^2 + \sigma_D^2\right] + p_x^2\sigma_x^2 + \left(p_{\hat{F}} Y_3\right)^2 \sigma_S^2 \\
&= \left(\rho^F\right)^2 \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \Sigma_0 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \sigma_F^2 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2 + p_x^2\sigma_x^2
\end{aligned}$$

Plug expression for V^I and V^U back into equation A.24 and take $\lambda \to 0$, rearranging, one obtains

$$\left(R - \rho^x\right) p_x = \alpha\left(\left(\rho^F\right)^2 \left(1 + \frac{\rho^F}{R - \rho^F}\left(\theta_0\right)\right)^2 \Sigma_0 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \sigma_F^2 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + p_x^2\sigma_x^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2\right)$$

Rearrange:

$$\alpha\sigma_x^2 p_x^2 - \left(R - \rho^x\right) p_x + \alpha\left(\left(\rho^F\right)^2 \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \Sigma_0 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \sigma_F^2 + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2\right) = 0$$

Thus, p_x is well defined if and only if

$$\left(R - \rho^x\right)^2 - 4\alpha^2\sigma_x^2\left(\left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right] + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2\right) > 0$$

Denote $\Delta = \left(R - \rho^x\right)^2 - 4\alpha^2\sigma_x^2\left(\left(1 + \frac{\rho^F}{R - \rho^F}\theta_0\right)^2 \left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right] + \left(1 + \frac{\rho^F}{R - \rho^F}\theta_2\right)^2 \sigma_D^2 + \left(\frac{\rho^F}{R - \rho^F}\theta_3\right)^2 \sigma_S^2\right)$, then at high volatility equilibria:

$$p_x = \frac{R - \rho^x + \sqrt{\Delta}}{2\alpha\sigma_x^2} \tag{A.31}$$

By continuity, when λ is very small, real roots to p_x still exists.

Auxiliary Results

Derivative of an inverse matrix

Proposition A.1 *Let A be a nonsingular, $m \times m$ matrix whose elements are functions of the scalar parameter α. Then:*

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1}\frac{\partial A}{\partial \alpha}A^{-1}$$

Proof. The definition of inverse is

$$A^{-1}A = I$$

Differentiating this expression with respect to α and rearranging, one obtains the result. ■

Auxiliary Derivatives

Before we get into the proof of lemma 3.3, we need to take derivatives with respect to $Y_1, Y_2, Y_3, e_1, e_2, e_3, e_4, e_5, e_6, e_7, V^I, V^U$ given in equation A.1, equations A.6 through A.13, and equations A.16, A.18 Note that all values are taken as $\lambda \to 0$.

1. Y_1, Y_2, Y_3

   Y_1, Y_2, Y_3 are the first row of the matrix $[A\Sigma A'G' + QG'] [GA\Sigma A'G' + GQG' + R]^{-1}$. Thus, when taking derivatives, we need to evaluate

$$\begin{aligned}\frac{\partial [A\Sigma A'G' + QG'] [GA\Sigma A'G' + GQG' + R]^{-1}}{\partial p_x} &= \frac{\partial [A\Sigma A'G' + QG']}{\partial p_x}\left[GA\Sigma A'G' + GQG' + R\right]^{-1} \\ &\quad + \left[A\Sigma A'G' + QG'\right]\frac{\partial [GA\Sigma A'G' + GQG' + R]^{-1}}{\partial p_x}\end{aligned}$$

   We need to evaluate each term separately:

$$\begin{aligned}A\Sigma A'G' + QG' &= \begin{bmatrix} p_F\left[\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right] - p_x\left[\rho^F\rho^x\frac{p_F}{p_x}\Sigma_F\right] & \left(\rho^F\right)^2 a_{11} + \sigma_F^2 & \left(\rho^F\right)^2 a_{11} + \sigma_F^2 \\ p_F\left[\rho^F\rho^x\frac{p_F}{p_x}\Sigma_F\right] - p_x\left[(\rho^x)^2\left(\frac{p_F}{p_x}\right)^2\Sigma_F + \sigma_x^2\right] & \rho^F\rho^x\frac{p_F}{p_x}\Sigma_F & \rho^F\rho^x\frac{p_F}{p_x}\Sigma_F \end{bmatrix} \\ &\to \begin{bmatrix} 0 & \left(\rho^F\right)^2 a_{11} + \sigma_F^2 & \left(\rho^F\right)^2 a_{11} + \sigma_F^2 \\ -p_x\sigma_x^2 & 0 & 0 \end{bmatrix} \text{ as } \lambda \to 0 \\ \frac{\partial [A\Sigma A'G' + QG']}{p_x} &\to \begin{bmatrix} \left(\rho^F\right)^2\Sigma_0 + \sigma_F^2 - \rho^F\rho^x\Sigma_0 & 0 & 0 \\ 0 & \rho^F\rho^x\frac{1}{p_x}\Sigma_0 & \rho^F\rho^x\frac{1}{p_x}\Sigma_0 \end{bmatrix} \\ \frac{\partial [A\Sigma A'G' + QG']}{p_x} &\to \begin{bmatrix} 0 & 0 & 0 \\ -\sigma_x^2 & 0 & 0 \end{bmatrix}\end{aligned}$$

$$\left[GA\Sigma A'G' + GQG' + R\right]^{-1} \to \begin{bmatrix} \frac{1}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2\right)}{\Theta} & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F + \sigma_F^2 + \sigma_D^2\right)}{\Theta} \end{bmatrix}$$

We use proposition A.1 to evaluate the following two derivatives:

$$
\begin{aligned}
\frac{\partial\left[GA\Sigma A'G'+GQG'+R\right]^{-1}}{\partial p_F} &= -\left[GA\Sigma A'G'+GQG'+R\right]^{-1}\frac{\partial\left[GA\Sigma A'G'+GQG'+R\right]}{\partial p_F}\left[GA\Sigma A'G'+GQG'+R\right]^{-1} \\
&= -\left[GA\Sigma A'G'+GQG'+R\right]^{-1} \\
&\quad \begin{bmatrix} 0 & \left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F & \left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F \\ \left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F & 0 & 0 \\ \left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F & 0 & 0 \end{bmatrix} \\
&\quad \left[GA\Sigma A'G'+GQG'+R\right]^{-1} \\
&= -\begin{bmatrix} 0 & \frac{\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F}{(p_x)^2\sigma_x^2} & \frac{\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F}{(p_x)^2\sigma_x^2} \\ \frac{(p_x)^2\sigma_x^2(\sigma_S^2)}{\Theta}\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F\right) & 0 & 0 \\ \frac{(p_x)^2\sigma_x^2(\sigma_D^2)}{\Theta}\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F\right) & 0 & 0 \end{bmatrix} \\
&\quad \begin{bmatrix} \frac{1}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_D^2\right)}{\Theta} \end{bmatrix} \\
&= -\begin{bmatrix} 0 & \frac{\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F}{(p_x)^2\sigma_x^2}\frac{(p_x)^2\sigma_x^2\sigma_S^2}{\Theta} & \frac{\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F}{(p_x)^2\sigma_x^2} \\ \frac{\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F}{(p_x)^2\sigma_x^2}\frac{(p_x)^2\sigma_x^2(\sigma_S^2)}{\Theta} & 0 & 0 \\ \frac{\left(\rho^F\right)^2\Sigma_F+\sigma_F^2-\rho^F\rho^x\Sigma_F}{(p_x)^2\sigma_x^2}\frac{(p_x)^2\sigma_x^2(\sigma_D^2)}{\Theta} & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\left[GA\Sigma A'G'+GQG'+R\right]^{-1}}{\partial p_x} &= -\left[GA\Sigma A'G'+GQG'+R\right]^{-1}\frac{\partial\left[GA\Sigma A'G'+GQG'+R\right]}{\partial p_x}\left[GA\Sigma A'G'+GQG'+R\right]^{-1} \\
&= -\left[GA\Sigma A'G'+GQG'+R\right]^{-1}\begin{bmatrix} 2p_x\sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left[GA\Sigma A'G'+GQG'+R\right]^{-1} \\
&= -\begin{bmatrix} \frac{1}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_D^2\right)}{\Theta} \end{bmatrix}\begin{bmatrix} 2p_x\sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\quad \begin{bmatrix} \frac{1}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_D^2\right)}{\Theta} \end{bmatrix} \\
&= -\begin{bmatrix} \frac{2p_x\sigma_x^2}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{(p_x)^2\sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2\right)}{\Theta} & \frac{(p_x)^2\sigma_x^2\left(\left(\rho^F\right)^2\Sigma_F+\sigma_F^2+\sigma_D^2\right)}{\Theta} \end{bmatrix} \\
&= -\begin{bmatrix} \frac{2p_x\sigma_x^2}{(p_x)^4\sigma_x^4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Thus,

$$\frac{\partial [Y_1, Y_2, Y_3]}{\partial p_F} = (\text{The first column of }) \begin{bmatrix} \left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 - \rho^F \rho^x \Sigma_0 & 0 & 0 \\ 0 & \rho^F \rho^x \frac{1}{p_x} \Sigma_0 & \rho^F \rho^x \frac{1}{p_x} \Sigma_0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{(p_x)^2 \sigma_x^2} & 0 & 0 \\ 0 & \frac{(p_x)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right)}{\Theta} & -\frac{(p_x)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)}{\Theta} \\ 0 & -\frac{(p_x)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)}{\Theta} & \frac{(p_x)^2 \sigma_x^2 \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right)}{\Theta} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & \left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 & \left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 \\ -p_x \sigma_x^2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F}{(p_x)^2 \sigma_x^2} \frac{(p_x)^2 \sigma_x^2 \sigma_S^2}{\Theta} & \frac{\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F}{(p_x)^2 \sigma_x^2} \frac{(p_x)^2 \sigma_x^2 \sigma_D^2}{\Theta} \\ \frac{\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F}{(p_x)^2 \sigma_x^2} \frac{(p_x)^2 \sigma_x^2 \left(\sigma_S^2\right)}{\Theta} & 0 & 0 \\ \frac{\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F}{(p_x)^2 \sigma_x^2} \frac{(p_x)^2 \sigma_x^2 \left(\sigma_D^2\right)}{\Theta} & 0 & 0 \end{bmatrix}$$

Thus,

$$\frac{\partial [Y_1, Y_2, Y_3]}{\partial p_F} \to \left[\frac{\left[\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 - \rho^F \rho^x \Sigma_0\right]}{(p_x)^2 \sigma_x^2} - \left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right) \frac{\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 - \rho^F \rho^x \Sigma_0}{(p_x)^2 \sigma_x^2} \frac{(p_x)^2 \sigma_x^2 \left(\sigma_S^2 + \sigma_D^2\right)}{\Theta}, 0, 0 \right]$$

Thus,

$$\begin{aligned} \frac{\partial Y_1}{\partial p_F} &= \frac{\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 - \rho^F \rho^x \Sigma_0}{(p_x)^2 \sigma_x^2} - \left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right) \frac{\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 - \rho^F \rho^x \Sigma_0}{(p_x)^2 \sigma_x^2} \frac{(p_x)^2 \sigma_x^2 \left(\sigma_S^2 + \sigma_D^2\right)}{\Theta} \\ &= \frac{\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2 - \rho^F \rho^x \Sigma_0}{(p_x)^2 \sigma_x^2} \left(1 - \frac{\left(\left(\rho^F\right)^2 \Sigma_0 + \sigma_F^2\right) \left(\sigma_S^2 + \sigma_D^2\right)}{\left[\left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_D^2\right) \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 + \sigma_S^2\right) - \left(\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right)^2\right]} \right) \end{aligned}$$

Similarly, one can show that

$$\frac{\partial [Y_1, Y_2, Y_3]}{\partial p_x} = [0, 0, 0]$$

2. $e_1 = \left(\frac{\rho^F}{R - \rho^F} - p_F\right) \rho^F \left(1 - Y_1 p_F - Y_2 - Y_3\right)$

$$\begin{aligned} e_1 &= \left(\frac{\rho^F}{R - \rho^F} - p_F\right) \rho^F \left(1 - Y_1 p_F - Y_2 - Y_3\right) \\ &\to \frac{\rho^F}{R - \rho^F} \rho^F \left(1 - \theta_0\right) \\ \frac{\partial e_1}{\partial p_F} &= -\rho^F \left(1 - Y_1 p_F - Y_2 - Y_3\right) + \left(\frac{\rho^F}{R - \rho^F} - p_F\right) \rho^F \left(1 - \frac{\partial Y_1}{\partial p_F} p_F - Y_1 - \frac{\partial Y_2}{\partial p_F} - \frac{\partial Y_3}{\partial p_F}\right) \\ &\to -\rho^F \left(1 - \theta_0\right) + \frac{\rho^F}{R - \rho^F} \rho^F \end{aligned}$$

3. $e_2 = p_{\hat{F}} Y_1 p_x \rho^x = \left(\frac{\rho^F}{R-\rho^F} - p_F\right) Y_1 p_x \rho^x$

$$\begin{aligned}
e_2 &\to 0 \\
\frac{\partial e_2}{\partial p_F} &= -Y_1 p_x \rho^x + \left(\frac{\rho^F}{R-\rho^F} - p_F\right) \frac{\partial Y_1 p_x}{\partial p_F} \rho^x \to 0 + 0 = 0 \\
\frac{\partial e_2}{\partial p_x} &= \left(\frac{\rho^F}{R-\rho^F} - p_F\right) Y_1 \rho^x + \left(\frac{\rho^F}{R-\rho^F} - p_F\right) \frac{\partial Y_1}{\partial p_x} p_x \rho^x \to 0 \\
\frac{\partial e_2}{\partial \Sigma_F} &= \left(\frac{\rho^F}{R-\rho^F} - p_F\right) \frac{\partial Y_1}{\partial \Sigma_F} p_x \rho^x = 0
\end{aligned}$$

4. $e_3 = \rho^F \left(1 + p_F + p_{\hat{F}} \left(Y_1 p_F + Y_2 + Y_3\right)\right)$

$$\begin{aligned}
e_3 &\to \rho^F \left(1 + \frac{\rho^F}{R-\rho^F} \theta_0\right) \\
\frac{\partial e_3}{\partial p_F} &= \rho^F \left(1 - (Y_1 p_F + Y_2 + Y_3) + \left(\frac{\rho^F}{R-\rho^F} - p_F\right) \left(\frac{\partial Y_2}{\partial p_F} + \frac{\partial Y_3}{\partial p_F}\right)\right) \to \rho^F (1 - \theta_0) \\
\frac{\partial e_3}{\partial p_x} &= \rho^F \left(p_{\hat{F}} \left(\frac{\partial Y_1}{\partial p_x} p_F + \frac{\partial Y_2}{\partial p_x} + \frac{\partial Y_3}{\partial p_F}\right)\right) = 0
\end{aligned}$$

5. $e_4 = \rho^x \left(p_{\hat{F}} Y_1 + 1\right) p_x$

$$e_4 \to \rho^x p_x$$

$$\begin{aligned}
\frac{\partial e_4}{\partial p_F} &= \rho_x \left(-Y_1 p_x + p_{\hat{F}} \frac{\partial Y_1 p_x}{\partial p_F}\right) = \rho_x p_x \frac{\rho^F}{R-\rho^F} \frac{\partial Y_1}{\partial p_F} \\
\frac{\partial e_4}{\partial p_x} &= \rho^x \left(p_{\hat{F}} Y_1 + 1\right) + \rho^x p_{\hat{F}} \frac{\partial Y_1}{\partial p_x} p_x \to \rho^x
\end{aligned}$$

6. $e_5 = 1 + p_F + p_{\hat{F}} \left(Y_1 p_F + Y_2 + Y_3\right)$

$$e_5 \to 1 + \frac{\rho^F}{R-\rho^F} \theta_0$$

$$\frac{\partial e_5}{\partial p_F} = 1 - (Y_1 p_F + Y_2 + Y_3) + p_{\hat{F}} \left(\frac{\partial Y_2}{\partial p_F} + \frac{\partial Y_3}{\partial p_F}\right) = 1 - Y_2 - Y_3 = 1 - \theta_0$$

$$\frac{\partial e_5}{\partial p_x} = p_{\hat{F}} \left(\frac{\partial Y_1}{\partial p_x} p_F + \frac{\partial Y_2}{\partial p_x} + \frac{\partial Y_3}{\partial p_x}\right) \to 0$$

7. $e_6 = \left(p_{\hat{F}} Y_1 p_x + p_x\right)$

$$e_6 \to p_x$$

$$\begin{aligned}
\frac{\partial e_6}{\partial p_F} &= \left(-Y_1 p_x + p_{\hat{F}} \frac{\partial Y_1}{\partial p_F} p_x\right) \to \frac{\rho^F}{R-\rho^F} \frac{\partial Y_1}{\partial p_F} p_x \\
\frac{\partial e_6}{\partial p_x} &= p_{\hat{F}} \frac{\partial Y_1}{\partial p_x} p_x + p_{\hat{F}} Y_1 + 1 = 1
\end{aligned}$$

8. $e_7 = p_{\hat{F}} Y_2 + 1$

$$\begin{aligned}
e_7 &\to \frac{\rho^F}{R-\rho^F} \theta_2 + 1 \\
\frac{\partial e_7}{\partial p_F} &= -Y_2 + p_{\hat{F}} \frac{\partial Y_2}{\partial p_F} \to -\theta_2 \\
\frac{\partial e_7}{\partial p_x} &= 0
\end{aligned}$$

9. $e_8 = p_{\hat{F}} Y_3$

$$\begin{aligned} e_8 &\rightarrow \frac{\rho^F}{R-\rho^F}\theta_3 \\ \frac{\partial e_8}{\partial p_F} &= -Y_3 + p_{\hat{F}}\frac{\partial Y_3}{\partial p_F} = -\theta_3 \\ \frac{\partial e_8}{\partial p_F} &= 0 \end{aligned}$$

10. V^I.

$$\begin{aligned} V^I &= e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2 + e_8^2\sigma_S^2 \rightarrow \left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)^2\sigma_F^2 + \left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\sigma_D^2 + \left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2 + p_x^2\sigma_x^2 \quad \text{(A.32)} \\ \frac{\partial V^I}{\partial p_F} &= 2e_5\frac{\partial e_5}{\partial p_F}\sigma_F^2 + 2e_6\frac{\partial e_6}{\partial p_F}\sigma_x^2 + 2e_7\frac{\partial e_7}{\partial p_F}\sigma_D^2 + 2e_8\frac{\partial e_8}{\partial p_F}\sigma_S^2 \\ &\rightarrow 2\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)(1-\theta_0)\sigma_F^2 + 2\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F}p_x^2\sigma_x^2 + 2\left(\frac{\rho^F}{R-\rho^F}\theta_2+1\right)(-\theta_2)\sigma_D^2 - 2\frac{\rho^F}{R-\rho^F}\theta_3^2\sigma_S^2 \quad \text{(A.33)} \\ \frac{\partial V^I}{\partial p_x} &= 2e_5\frac{\partial e_5}{\partial p_x}\sigma_F^2 + 2e_6\frac{\partial e_6}{\partial p_x}\sigma_x^2 + 2e_7\frac{\partial e_7}{\partial p_x}\sigma_D^2 + 2e_8\frac{\partial e_8}{\partial p_x}\sigma_S^2 \\ &\rightarrow 2e_6\frac{\partial e_6}{\partial p_x}\sigma_x^2 = 2p_x\sigma_x^2 \end{aligned}$$

11. V^U :

$$\begin{aligned} V^U &= H\Sigma H' + V^I \\ &= \left(e_3 + \frac{p_F}{p_x}e_4\right)^2\Sigma_F + V^I \\ &= \left(\rho^F\left(1+p_F+p_{\hat{F}}\left(Y_1p_F+Y_2+Y_3\right)\right) - \frac{p_F}{p_x}\rho^x\left(p_{\hat{F}}Y_1p_x+p_x\right)\right)^2\Sigma_F + V^I \\ &\rightarrow \left(\rho^F\left(1+\frac{\rho^F}{R-\rho^F}\theta_0\right)\right)^2\Sigma_0 + V^I \end{aligned}$$

Thus,

$$V^U = \left(\rho^F + \rho^F\frac{\rho^F}{R-\rho^F}y_0\right)^2\Sigma_F + \left(1+\frac{\rho^F}{R-\rho^F}\theta_2\right)^2\left[\sigma_F^2+\sigma_D^2\right] + \left(\frac{\rho^F}{R-\rho^F}\theta_3\right)^2\sigma_S^2 + p_x^2\sigma_x^2 \quad \text{(A.34)}$$

$$\frac{\partial V^U}{\partial p_F} \rightarrow 2\left(\rho^F + \rho^F\frac{\rho^F}{R-\rho^F}\theta_0\right)\left(\rho^F - \rho^F\theta_0 - \rho^x\right)\Sigma_0 + \frac{\partial V^I}{\partial p_F} \quad \text{(A.35)}$$

$$\begin{aligned} \frac{\partial V^U}{\partial p_x} &= 2\left(\rho^F + \rho^F\frac{\rho^F}{R-\rho^F}Y_2 + \left(\rho^F - \rho^F Y_2 - \rho^x\right)p_F\right)\left(\rho^F\frac{\rho^F}{R-\rho^F}\frac{\partial Y_2}{\partial p_x} - \rho^F p_F\frac{\partial Y_2}{\partial p_x}\right)\Sigma_F + \frac{\partial V^I}{\partial p_x} \\ &\rightarrow \frac{\partial V^I}{\partial p_x} = 2p_x\sigma_x^2 \end{aligned}$$

A.1 Proof of lemma 3.3

The proof is organized into two parts. The first part proves that the value of information is the ratio of uncertainty faced by the uninformed and informed and derives an exact formula for $\pi(\lambda)$. The second part derives the derivative $\pi'(\lambda)$ when $\lambda \rightarrow 0$.

Step 1: Deriving formula of $\pi(\lambda)$ We will show that, at the exogenous-information steady state $\Phi(\lambda)$, the value of information

$$\pi(\lambda) = \frac{W^U}{W^I} = \sqrt{\frac{Var(P' + D'|\Omega^U)}{Var(P' + D'|\Omega^I)}}$$

This is an extension of Theorem 2 in Grossman and Stiglitz (1980). Note that P' denotes the next period equilibrium price whereas P denotes current price. Simplify agents' budget constraint: $c(s') = (D' + P' - RP)e$. Plugging into the utility function, we obtain the expected utility of each type of agent after the market opens:

$$W^i(P) = \max_e \int_{s'} U((D' + P' - RP)e)dH(s'|\Omega^i)$$

Given CARA utility:

$$\begin{aligned}
W^i(P) &= \max_e \int_{s'} U((D' + P' - RP)e)dH(s'|\Omega^i) \\
&= \max_e \int_{s'} -e^{(-\alpha((D'+P'-RP)e))}dH(s'|\Omega^i) \\
&= \max_e -\exp[E[-\alpha((D' + P' - RP)e)|I^i] + \frac{1}{2}Var(-\alpha((D' + P' - RP)e)|\Omega^i)] \\
&= \max_e -\exp[-\alpha(E[D' + P' - RP|\Omega^i]e - \frac{1}{2}\alpha e^2 Var(D' + P' - RP|\Omega^i))]
\end{aligned} \tag{A.36}$$

Hence, maximizing over the objective function is equivalent to maximizing

$$\max_e E[D' + P' - RP|\Omega^i]e - \frac{1}{2}\alpha e^2 Var(D' + P' - RP|\Omega^i)$$

Solve for optimal s^*:

$$e^* = \frac{E[D' + P' - RP|\Omega^i]}{\alpha Var(D' + P' - RP|\Omega^i)}$$

Plug back into the original objective function:

$$\begin{aligned}
W^i(P) &= -\exp[-\frac{1}{2}\alpha\frac{(E[D' + P' - RP|\Omega^i])^2}{\alpha Var(D' + P' - RP|\Omega^i)}] \\
&= -\exp[-\frac{1}{2}\frac{(E[D' + P'|\Omega^i] - RP)^2}{Var(D' + P'|\Omega^i)}]
\end{aligned} \tag{A.37}$$

where the second equation follows because P is realized at this stage. Let

$$h = Var(D' + P'|\Omega^U) - Var(D' + P'|\Omega^I) > 0$$

The reason why it is greater than 0 is that the uninformed have residual uncertainty over the current F whereas the informed are perfectly informed about F. Taking the conditional expectation of the informed $W_I(P)$ of the uninformed agents' information set:

$$\begin{aligned}
E[W^i(P)|\Omega^U] &= E[-e^{-\frac{1}{2}\frac{(E[D'+P'|\Omega^I]-RP)^2}{Var(D'+P'|\Omega^I)}}|\Omega^U] \\
&= E[-e^{-\frac{1}{2}\frac{(E[D'+P'|\Omega^I]-RP)^2}{h}\frac{h}{Var(D'+P'|\Omega^I)}}|\Omega^U] \\
&= E[-e^{-\frac{1}{2}\frac{h}{Var(D'+P'|\Omega^I)}z^2}|\Omega^U],
\end{aligned} \tag{A.38}$$

where $z = \frac{(E[D'+P'|\Omega^I]-RP)}{\sqrt{h}}$.

Thus, by the moment-generating function of a noncentral chi-squared distribution (formula $A21$ of Grossman and Stiglitz (1980)):

$$
\begin{aligned}
E[W^i(P)|\Omega^U] &= \frac{1}{\sqrt{1+\frac{h}{Var(D'+P'|\Omega^I)}}}\exp(\frac{-E[z|\Omega^U]^2\frac{1}{2}\frac{h}{Var(D'+P'|\Omega^I)}}{1+\frac{h}{Var(D'+P'|\Omega^I)}}) \\
&= \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|I^U)}}\exp(\frac{-E[z|\Omega^U]^2\frac{1}{2}\frac{h}{Var(D'+P'|\Omega^i)}}{1+\frac{h}{Var(D'+P'|\Omega^i)}}) \\
&= \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^U)}}\exp(\frac{-\frac{1}{2}(E[D'+P'|\Omega^U]-RP)^2}{Var(D'+P'|\Omega^U)}) \\
&= \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^U)}}W_U(P)
\end{aligned}
$$

Integrating on both sides with respect to the current state s, one gets:

$$
W_I = \sqrt{\frac{Var(D'+P'|\Omega^I)}{Var(D'+P'|\Omega^U)}}W_U \tag{A.39}
$$

Thus,

$$
\pi(\lambda) = \frac{W_U}{W_I} = \sqrt{\frac{Var(D'+P'|\Omega^U)}{Var(D'+P'|\Omega^I)}}
$$

Now we know that

$$
D'+P' = a + e_1\hat{F} + e_2\hat{x} + e_3F - e_4x + e_5\varepsilon^{F\prime} - e_6\varepsilon^{x\prime} + e_7\varepsilon^{D\prime}, \tag{A.40}
$$

where expressions of e_i are given by A.6 through A.13.

Thus, by equation A.17 and A.19:

$$
\begin{aligned}
\pi(\lambda) = \frac{W_U}{W_I} &= \sqrt{\frac{Var(D'+P'|\Omega^U)}{Var(D'+P'|\Omega^I)}} \\
&= \sqrt{\frac{H\Sigma H' + e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2}{e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2}}
\end{aligned} \tag{A.41}
$$

Note that all the objects in A.41: H, Σ, e_5, e_6, e_7, are functions of Σ_F, p_F, p_x, which are ultimately implicit functions of λ, determined by equations A.2, A.21, and A.22. I omit the dependence here just to ease notation.

Step 2: Evaluate $\pi'(\lambda)$ when $\lambda \to 0$ First we represent equations A.2, A.21, and A.22 as

$$\begin{aligned} G_1(\Sigma_F, p_F, p_x) &= 0 \\ G_2(\Sigma_F, p_F, p_x, \lambda) &= 0 \\ G_3(\Sigma_F, p_F, p_x, \lambda) &= 0 \end{aligned}$$

from which we can derive the derivates $\frac{\partial \Sigma_F}{\partial \lambda}$, $\frac{\partial p_F}{\partial \lambda}$ and $\frac{\partial p_x}{\partial \lambda}$ by implicit differentiation. To begin, note that function G_1 does not contain λ. Therefore, we can think of Σ_F as an implicit function of p_F and p_x and evaluate the implicit derivative $\frac{\partial \Sigma_F}{\partial p_F}$ and $\frac{\partial \Sigma_F}{\partial p_x}$.We can show the following important result: $\frac{\partial \Sigma_F}{\partial p_F} = \frac{\partial \Sigma_F}{\partial p_x} = 0$:

$$\begin{aligned} G_1(\Sigma_F, p_F, p_x) &= \left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 p_F^2 \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[p_F^2 \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - p_x^2 \sigma_x^2\right]}{\sigma_D^2 \left[p_F^2 \left(\rho^F - \rho^x\right)^2 \Sigma_F + p_F^2 \sigma_F^2 + p_x^2 \sigma_x^2\right] + \sigma_F^2 \left(\rho^F\right)^2 p_F^2 \Sigma_F + \left(\rho^F\right)^2 p_x^2 \sigma_x^2 \Sigma_F + p_x^2 \sigma_x^2 \sigma_F^2} - \Sigma_F \\ &= \left[\left(\rho^F\right)^2 - 1\right] \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 \frac{p_F^2}{p_x^2} \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[\frac{p_F^2}{p_x^2} \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - \sigma_x^2\right]}{\sigma_D^2 \left[\frac{p_F^2}{p_x^2} \left(\rho^F - \rho^x\right)^2 \Sigma_F + \frac{p_F^2}{p_x^2} \sigma_F^2 + \sigma_x^2\right] + \sigma_F^2 \left(\rho^F\right)^2 \frac{p_F^2}{p_x^2} \Sigma_F + \left(\rho^F\right)^2 \sigma_x^2 \Sigma_F + \sigma_x^2 \sigma_F^2} \\ &= \left[\left(\rho^F\right)^2 - 1\right] \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 u \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[u \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - \sigma_x^2\right]}{\sigma_D^2 \left[u \left(\rho^F - \rho^x\right)^2 \Sigma_F + u \sigma_F^2 + \sigma_x^2\right] + \sigma_F^2 \left(\rho^F\right)^2 \Sigma_F u + \left(\rho^F\right)^2 \sigma_x^2 \Sigma_F + \sigma_x^2 \sigma_F^2} \\ &\quad \text{where } u = \left[\frac{p_F^2}{p_x^2}\right]^2 \end{aligned}$$

Define a new function:

$$N_1(\Sigma_F, u) = \left[\left(\rho^F\right)^2 - 1\right] \Sigma_F + \sigma_F^2 - \frac{\left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2 - \rho^F \rho^x \Sigma_F\right]^2 u \sigma_D^2 - \left[\left(\rho^F\right)^2 \Sigma_F + \sigma_F^2\right]^2 \left[u \Sigma_F \left(\rho^F - \rho^x\right) \rho^x - \sigma_x^2\right]}{\sigma_D^2 \left[u \left(\rho^F - \rho^x\right)^2 \Sigma_F + u \sigma_F^2 + \sigma_x^2\right] + \sigma_F^2 \left(\rho^F\right)^2 \Sigma_F u + \left(\rho^F\right)^2 \sigma_x^2 \Sigma_F + \sigma_x^2 \sigma_F^2}$$

Thus,

$$G_1(\Sigma_F, p_F, p_x) = N_1\left(\Sigma_F, \frac{p_F^2}{p_x^2}\right)$$

By implicit differentiation:

$$\frac{\partial \Sigma_F}{\partial p_F} = -\frac{\frac{\partial G_1}{\partial p_F}}{\frac{\partial G_1}{\partial \Sigma_F}} - \frac{\frac{\partial N_1}{\partial u} \frac{\partial u}{\partial p_F}}{\frac{\partial N_1}{\partial \Sigma_F}} = -\frac{\frac{\partial N_1}{\partial u}}{\frac{\partial N_1}{\partial \Sigma_F}} 2 \frac{p_F}{p_x^2} \to 0$$

Likewise,

$$\frac{\partial \Sigma_F}{\partial p_x} \to 0$$

Now, G_2 and G_3 are given by

$$G_2 = \lambda \frac{e_3 - e_2 \frac{p_F}{p_x}}{\alpha V^I} - \left(\lambda \frac{R}{\alpha V^I} + (1 - \lambda) \frac{R}{\alpha V^U}\right) p_F = 0$$

$$G_3 = \left[\lambda \frac{1}{\alpha V^I} + (1 - \lambda) \frac{1}{\alpha V^U}\right] \left[-e_4 + e_2 + R p_x\right] - 1 = 0$$

To derive $\frac{\partial p_F}{\partial \lambda}$ and $\frac{\partial p_x}{\partial \lambda}$, define $\pi = \frac{V^U}{V^I}$. Substitute in π whenever possible. Then,

$$
\begin{aligned}
G_2(\lambda, p_F, p_x, \pi) &= \lambda \frac{e_3 - e_2 \frac{p_F}{p_x}}{\alpha} \pi - \left(\lambda \frac{1}{\alpha} \pi + (1-\lambda) \frac{1}{\alpha} \right) R p_F = 0 \\
G_3(\lambda, p_F, p_x, \pi) &= \left[\frac{\lambda}{\alpha} \pi + (1-\lambda) \frac{1}{\alpha} \right] [-e_4 + e_2 + R p_x] - V^U = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial G_2}{\partial \lambda} &= \frac{e_3 - e_2 \frac{p_F}{p_x}}{\alpha} \pi \rightarrow \frac{\rho^F \left(1 + \frac{\rho^F}{R - \rho^F} \theta_0 \right)}{\alpha} \pi \\
\frac{\partial G_2}{\partial p_F} &= \lambda \frac{\frac{\partial e_3 - e_2 \frac{p_F}{p_x}}{\partial p_F}}{\alpha} \pi - \left(\lambda \frac{R}{\alpha} \pi + (1-\lambda) \frac{R}{\alpha} \right) = -\frac{R}{\alpha} \\
\frac{\partial G_2}{\partial p_x} &= 0 \\
\frac{\partial G_2}{\partial \pi} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial G_3}{\partial \lambda} &= \left[\frac{1}{\alpha} \pi - \frac{1}{\alpha} \right] [-e_4 + e_2 + R p_x] \rightarrow \frac{1}{\alpha} [\pi - 1] [R - \rho^x] p_x \\
\frac{\partial G_3}{\partial p_F} &= \left[\frac{\lambda}{\alpha} \pi + (1-\lambda) \frac{1}{\alpha} \right] \left[\frac{\partial (-e_4 + e_2)}{\partial p_F} \right] - \frac{\partial V^U}{\partial p_F} = -\frac{1}{\alpha} \rho_x p_x \frac{\rho^F}{R - \rho^F} \frac{\partial Y_1}{\partial p_F} - \frac{\partial V^U}{\partial p_F} \\
\frac{\partial G_3}{\partial p_x} &= \left[\frac{\lambda}{\alpha} \pi + (1-\lambda) \frac{1}{\alpha} \right] \left[\frac{\partial (-e_4 + e_2)}{\partial p_x} + R \right] - \frac{\partial V^U}{\partial p_x} = \frac{1}{\alpha} [-\rho^x + R] - \frac{\partial V^U}{\partial p_x} \\
\frac{\partial G_3}{\partial \pi} &= 0
\end{aligned}
$$

Total differentiation of G_2 and G_3 gives

$$
\begin{bmatrix} \frac{\partial G_2}{\partial p_F} + \frac{\partial G_2}{\partial \pi} \frac{\partial \pi}{\partial p_F} & \frac{\partial G_2}{\partial p_x} + \frac{\partial G_2}{\partial \pi} \frac{\partial \pi}{\partial p_x} \\ \frac{\partial G_3}{\partial p_F} + \frac{\partial G_3}{\partial \pi} \frac{\partial \pi}{\partial p_F} & \frac{\partial G_3}{\partial p_x} + \frac{\partial G_3}{\partial \pi} \frac{\partial \pi}{\partial p_x} \end{bmatrix} \begin{bmatrix} \frac{\partial p_F}{\partial \lambda} \\ \frac{\partial p_x}{\partial \lambda} \end{bmatrix} = - \begin{bmatrix} \frac{\partial G_2}{\partial \lambda} \\ \frac{\partial G_3}{\partial \lambda} \end{bmatrix}
$$

Substituting in each term gives

$$
\begin{bmatrix} -\frac{R}{\alpha} & 0 \\ -\frac{1}{\alpha} \rho_x p_x \frac{\rho^F}{R - \rho^F} \frac{\partial Y_1}{\partial p_F} - \frac{\partial V^U}{\partial p_F} & \frac{1}{\alpha} [-\rho^x + R] - \frac{\partial V^U}{\partial p_x} \end{bmatrix} \begin{bmatrix} \frac{\partial p_F}{\partial \lambda} \\ \frac{\partial p_x}{\partial \lambda} \end{bmatrix} = - \begin{bmatrix} \frac{\rho^F \left(1 + \frac{\rho^F}{R - \rho^F} \theta_0 \right)}{\alpha} \pi \\ \frac{1}{\alpha} [\pi - 1] [R - \rho^x] p_x \end{bmatrix}
$$

$$
\begin{aligned}
\frac{\partial p_F}{\partial \lambda} &= \frac{\rho^F \left(1 + \frac{\rho^F}{R - \rho^F} \theta_0 \right)}{R} \pi \\
\frac{\partial p_x}{\partial \lambda} &= \frac{-\frac{1}{\alpha} [\pi - 1] [R - \rho^x] p_x + \left(\frac{1}{\alpha} \rho_x p_x \frac{\rho^F}{R - \rho^F} \frac{\partial Y_1}{\partial p_F} + \frac{\partial V^U}{\partial p_F} \right) \frac{\partial p_F}{\partial \lambda}}{\frac{1}{\alpha} [-\rho^x + R] - \frac{\partial V^U}{\partial p_x}}
\end{aligned} \tag{A.42}
$$

We can also evaluate

$$
\begin{aligned}
\frac{\partial \Sigma_F}{\partial \lambda} &= \frac{\partial \Sigma_F}{\partial p_F} \frac{\partial p_F}{\partial \lambda} + \frac{\partial \Sigma_F}{\partial p_x} \frac{\partial p_x}{\partial \lambda} \\
&= 0 \frac{\partial p_F}{\partial \lambda} + 0 \frac{\partial p_x}{\partial \lambda} = 0
\end{aligned}
$$

We are ready to evaluate $\pi'(\lambda)$ when $\lambda \to 0$.

Define

$$\Pi\left(\Sigma_F, p_F, p_x\right) = \frac{H\Sigma H + e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2}{e_5^2\sigma_F^2 + e_6^2\sigma_x^2 + e_7^2\sigma_D^2}$$

Then,

$$\pi(\lambda) = \sqrt{\Pi\left(\Sigma_F(\lambda), p_F(\lambda), p_x(\lambda)\right)}$$

$$\begin{aligned}
\pi'(\lambda) &= \frac{1}{2}\Pi^{-\frac{1}{2}}\left[\frac{\partial \Pi}{\partial \Sigma_F}\frac{\partial \Sigma_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\right] \\
&= \frac{1}{2}\Pi^{-\frac{1}{2}}\left[\frac{\partial \Pi}{\partial \Sigma_F}0 + \frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\right] \\
&= \frac{1}{2}\Pi^{-\frac{1}{2}}\left[\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}\right]
\end{aligned}$$

Thus, the sign of $\pi'(\lambda)$ only depends on the sign of $\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda}$, as $\Pi^{-\frac{1}{2}} > 0$.

$$\begin{aligned}
&\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda} \\
= \;& \frac{V^U\frac{\partial V^I}{\partial p_F} - V^I\frac{\partial V^U}{\partial p_F}}{\left(V^I\right)^2}\frac{\partial p_F}{\partial \lambda} + \frac{V^U\frac{\partial V^I}{\partial p_x} - V^I\frac{\partial V^U}{\partial p_x}}{\left(V^I\right)^2}\frac{\partial p_x}{\partial \lambda} \\
= \;& \frac{1}{\left(V^I\right)^2}\left\{\left[V^I\frac{\partial V^U}{\partial p_F} - V^U\frac{\partial V^I}{\partial p_F}\right]\frac{\partial p_F}{\partial \lambda} + \left[V^I\frac{\partial V^U}{\partial p_x} - V^U\frac{\partial V^I}{\partial p_x}\right]\frac{\partial p_x}{\partial \lambda}\right\} \\
= \;& \frac{1}{\left(V^I\right)^2}\left\{\left[V^I\frac{\partial V^U}{\partial p_F} - V^U\frac{\partial V^U}{\partial p_F} + V^U\left(\frac{\partial V^U}{\partial p_F} - \frac{\partial V^I}{\partial p_F}\right)\right]\frac{\partial p_F}{\partial \lambda} + \left[V^I - V^U\right]\frac{\partial V^U}{\partial p_x}\frac{-\frac{1}{\alpha}\left[\pi - 1\right]\left[R - \rho^x\right]p_x + \left(\frac{1}{\alpha}\rho_x p_x\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F} + \frac{\partial V^U}{\partial p_F}\right)\frac{\partial p_F}{\partial \lambda}}{\frac{1}{\alpha}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x}}\right. \\
= \;& \frac{1}{\left(V^I\right)^2}\left\{V^U\left(\frac{\partial V^U}{\partial p_F} - \frac{\partial V^I}{\partial p_F}\right)\frac{\partial p_F}{\partial \lambda} + \left[V^I - V^U\right]\left[\frac{\partial V^U}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial V^U}{\partial p_x}\frac{-\frac{1}{\alpha}\left[\pi - 1\right]\left[R - \rho^x\right]p_x + \left(\frac{1}{\alpha}\rho_x p_x\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F} + \frac{\partial V^U}{\partial p_F}\right)\frac{\partial p_F}{\partial \lambda}}{\frac{1}{\alpha}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x}}\right]\right\} \\
= \;& \frac{1}{\left(V^I\right)^2}\left\{V^U\left(\frac{\partial V^U}{\partial p_F} - \frac{\partial V^I}{\partial p_F}\right)\frac{\partial p_F}{\partial \lambda} + \left[V^I - V^U\right]\left[\frac{\frac{\partial V^U}{\partial p_F}\frac{\partial p_F}{\partial \lambda}\frac{1}{\alpha}\left[-\rho^x + R\right] - \frac{1}{\alpha}\frac{\partial V^U}{\partial p_x}\left[\pi - 1\right]\left[R - \rho^x\right]p_x + \frac{\partial V^U}{\partial p_x}\frac{1}{\alpha}\rho_x p_x\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F}\frac{\partial p_F}{\partial \lambda}}{\frac{1}{\alpha}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x}}\right]\right\} \\
= \;& \frac{1}{\left(V^I\right)^2}\left\{V^U\left(\frac{\partial V^U}{\partial p_F} - \frac{\partial V^I}{\partial p_F}\right)\frac{\partial p_F}{\partial \lambda} + \left[V^I - V^U\right]\left[\frac{\frac{\partial V^U}{\partial p_F}\frac{\partial p_F}{\partial \lambda}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x}\left[\pi - 1\right]\left[R - \rho^x\right]p_x + \frac{\partial V^U}{\partial p_x}\rho_x p_x\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F}\frac{\partial p_F}{\partial \lambda}}{\left[-\rho^x + R\right] - \alpha\frac{\partial V^U}{\partial p_x}}\right]\right\} \\
= \;& \frac{1}{\left(V^I\right)^2}\{V^U\left(2\left(\rho^F + \rho^F\frac{\rho^F}{R-\rho^F}\theta_0\right)\left(\rho^F - \rho^F\theta_0 - \rho^x\right)\Sigma_0\right)\frac{\partial p_F}{\partial \lambda} \\
&+ \left[V^I - V^U\right]\left[\frac{\frac{\partial V^U}{\partial p_F}\frac{\partial p_F}{\partial \lambda}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x}\left[\pi - 1\right]\left[R - \rho^x\right]p_x + \frac{\partial V^U}{\partial p_x}\rho_x p_x\frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F}\frac{\partial p_F}{\partial \lambda}}{\left[-\rho^x + R\right] - \alpha\frac{\partial V^U}{\partial p_x}}\right]\}
\end{aligned}$$

Thus, $\frac{\partial \Pi}{\partial p_F}\frac{\partial p_F}{\partial \lambda} + \frac{\partial \Pi}{\partial p_x}\frac{\partial p_x}{\partial \lambda} > 0$ if and only if

$$\left(1 - \theta_0\right)\rho^F > \rho^x + \phi(\rho^F, \rho^x, \sigma_x^2, \sigma_F^2, \sigma_D^2, \sigma_S^2, \alpha, R) \tag{A.43}$$

where

$$\phi(\rho^F, \rho^x, \sigma_x^2, \sigma_F^2, \sigma_D^2, \sigma_S^2, \alpha, R) = -\frac{\left[V^I - V^U\right]\left[\frac{\partial V^U}{\partial p_F}\frac{\partial p_F}{\partial \lambda}\left[-\rho^x + R\right] - \frac{\partial V^U}{\partial p_x}\left[\pi - 1\right]\left[R - \rho^x\right]p_x + \frac{\partial V^U}{\partial p_x}\rho_x p_x \frac{\rho^F}{R-\rho^F}\frac{\partial Y_1}{\partial p_F}\frac{\partial p_F}{\partial \lambda}\right]}{2V^U\left(\rho^F + \rho^F\frac{\rho^F}{R-\rho^F}\theta_0\right)\Sigma_0\frac{\partial p_F}{\partial \lambda}\left(-\rho^x + R - \alpha\frac{\partial V^U}{\partial p_x}\right)} \tag{A.44}$$

where V^I is given by A.32;V^U is given by A.34; $\frac{\partial p_F}{\partial \lambda}$ is given byA.42; $\frac{\partial V^U}{\partial p_F}$ is given by A.35; p_x is given by A.31.