Dynamic Information Acquisition and Time-Varying Uncertainty

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Abstract

This paper studies dynamic information acquisition in an infinite-horizon noisy rational expectations economy, with an emphasis on how a dynamic information market interacts with time-varying uncertainty. I prove a “one-shot” theorem that highlights the role of stock supply shock: If the supply shock is serially uncorrelated, the impact of exogenous uncertainty shocks can only last for one period. I then develop a recursive method to solve for dynamics with persistent stock supply, and find that information acquisition accommodates uncertainty shocks, but its effectiveness decreases with more persistent stock supply. Endogenous uncertainty shocks may arise due to a dynamic coordination failure in the information market.

Keywords: Information acquisition; Financial markets; Dynamic complementarity; Multiplicity

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1 Introduction

Market volatility fluctuates. To the extent that stock prices convey useful information about stock fundamentals, fluctuating volatility could translate into time-varying uncertainty that financial market investors must face. A natural way to counter this time-varying uncertainty is information acquisition. In a dynamic world, information acquisition could feed back into market volatility by altering investors’ trading behavior. Understanding the dynamic interplay between information acquisition and uncertainty is important, and allows us to answer a series of interesting questions. For example, does dynamic information acquisition necessarily accommodate uncertainty shocks? If so, what determines its effectiveness? Could it introduce additional fragility into the financial system?

I examine this issue by embedding the static information-acquisition model of Grossman and Stiglitz (1980) into a dynamic noisy rational expectations framework (as in Wang 1994; Spiegel 1998; Watanabe 2008). There is a long-lived stock that pays a dividend each period. The dividend is stochastic and consists of a persistent component (the stock fundamental) and a noisy component. The stock’s supply follows some mean-reverting process. Overlapping generations of investors, upon their birth, freely observe the entire history of stock prices, dividends, and public signals. They are then offered an opportunity to become informed, i.e., to observe the history of the stock fundamental at a cost.

To understand the nature of dynamic coordination in information markets, the paper starts with steady-state analysis. It is well known that overlapping-generation models with asymmetric information typically exhibit two financial market equilibria with different levels of stock market volatility (e.g. Spiegel, 1998; Watanabe, 2008). I show that conditional on financial market equilibrium, multiple steady states can arise in the information market. This is due to a dynamic complementarity in information acquisition. When investors make information decisions regarding the stock fundamental, their incentives to do so depend on how sensitive the future resale stock price is with respect to the fundamental. This creates a dynamic link across different generations of investors: If there were more informed investors in the future, the future resale stock price would become more sensitive to the fundamental and render information acquisition more attractive today. This dynamic complementarity effect may dominate the classic static substitutability effect in Grossman and Stiglitz (1980) and lead to an upward-sloping value of information in the share of
steady-state informed investors. This implies that multiple steady states can arise with appropriate levels of information cost.

I then seek to understand model dynamics. Given the steady-state multiplicity and the associated difficulty in solving for transition paths using the shooting method, I focus on recursive equilibria to explore model dynamics. The crucial observation is that the entire history of stock prices, dividends, and public signals can be summarized in a small set of two state variables: prior fundamental uncertainty and informative ratio of the last period, defined as the ratio of price sensitivity of fundamental and supply. The last-period informative ratio is part of the state, in addition to prior uncertainty, because it affects the updating strength of uninformed investors upon observing the current price signal, and therefore affects how investors’ conditional beliefs evolve. Given this observation, I propose a notion of recursive linear equilibrium (RLE) in which the pricing coefficients are time-invariant functions of the state variables. Pricing coefficients are allowed to vary over time, but only in a systematic way as the payoff-relevant state variables vary.

I characterize such RLE and explore factors that determine the effectiveness of endogenous information acquisition in accommodating exogenous uncertainty shocks. One of the key insights of the paper is that the persistence of stock supply serves crucial role. To illustrate I first prove a benchmark “one-shot” theorem: if the stock supply shock is serially uncorrelated, the impact of uncertainty shocks can only last for one period for any combinations of other model parameters. Upon heightened uncertainty, the value of information naturally rises, raising the share of investors acquiring information and reducing market uncertainty in the next period. The theorem states that, when stock supply is serially uncorrelated, such a response is so effective that the economy reverts right back to the original steady state in the next period. Roughly speaking, the dynamic model effectively collapses to the static Grossman-Stiglitz model despite the nontrivial two-way feedbacks between asset prices and information choices in the presence of long-lived assets. This serves as a powerful benchmark where endogenous information is extremely effective in accommodating uncertainty shocks in the presence of serially uncorrelated supply shocks.

In the case in which supply shocks are persistent, I numerically solve for the RLE and show that uncertainty shocks have long-lasting effects on the economy. The reason is that a more

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1Unlike models with dispersed information, the infinite regress problem does not arise here because, as in Grossman and Stiglitz (1980), the model admits a hierarchical information structure: Some investors are strictly more informed than others, and thus can perfectly infer their behavior.
persistent stock supply makes information acquisition less attractive, and thus less responsive to uncertainty shocks. To see this, consider the scenario in which there is a large exogenous but imperfectly observable supply shock hits the economy — say because there is a recession, and liquidity-constrained agents need cash. This would cause a low stock price to prevail. Investors would like to identify whether this low stock price is due to a bad fundamental or mispricing, as they could presumably step in and make a profit from the underpriced stock. However, if the liquidity shock is very persistent, the stock would be persistently underpriced. This implies that investors are unlikely to make a profit even if they have identified the underpriced stock, as the future resale stock price is likely to be low as well. This makes information acquisition less attractive. As investors are reluctant to acquire information, heightened uncertainty persists into the future.

The nonstationary dynamics of the model yield implications that cannot arise in steady-state environments. For example, in steady states stock fundamental uncertainty is closely tied to price informativeness: A more informative steady-state stock price always imply less fundamental uncertainty. This is not true when the economy is away from the steady states, because current fundamental uncertainty is determined jointly by current price informativeness and prior uncertainty. In fact, high quality of information and high uncertainty always coexist along transition paths following uncertainty shocks. Thus, current fundamental uncertainty can be high because of high fundamental uncertainty inherited from the past, despite a very informative current stock price.

So far I have analyzed the role of information choices in accommodating exogenous uncertainty shocks. However, the dynamic information market has its own “dark” side, as it may be a source of endogenous uncertainty shock due to the presence of dynamic complementarity in information acquisition. The same dynamic complementarity force that leads to steady-state multiplicity also leads to multiple RLEs that converge to these steady states with different levels of price informativeness. There always exist multiple RLE converging to steady states with different level of stock price informativeness. Specifically, there is an optimistic RLE in which most investors choose to acquire information because they expect future generations of investors to do so. There is also a pessimistic RLE where few investors choose to acquire information, because they expect few investors to acquire information in the future. Swings in investors’ beliefs act as a source of endogenous uncertainty shock. Given a history of trading and information choices, the economy can jump abruptly from one recursive equilibrium to another, creating additional volatility. Following
the belief shock, information choice and price informativeness switch abruptly, whereas fundamental uncertainty evolves slowly and converges to the new steady state. Consistent with Avdis (2016), I find that multiplicity is more likely to arise when the stock supply is less persistent.

Overall my analysis indicates that endogenous information acquisition acts as a double-edged sword: it serves as an accommodation device against exogenous uncertainty shocks, but may itself subject to dynamic coordination failure, introducing additional uncertainty into the financial system. In this regard, my analysis highlights the “mixed” role played by stock supply shocks. On the one hand, a more persistent stock supply makes endogenous uncertainty shock less likely to arise. On the other hand, it also makes information choices less responsive to exogenous uncertainty shocks. To the extent that stock supply shocks are at least partially driven by liquidity shocks to market participants — and thus regulators may affect stock supply by subsidizing cash-constrained agents — this creates a policy tradeoff regarding the optimal degree of supply persistence. If investors’ beliefs are stable and there are frequent exogenous variations in uncertainty, it is desirable to make the stock supply less persistent. If large swings in investors’ beliefs are likely, it might be desirable to make the supply shock very persistent so that the dynamic complementarity in information acquisition does not prevail.

**Literature Review**

This paper is related to the literature that studies information multiplicity in noisy rational expectation models (e.g. Veldkamp 2006a,b, Ganguli and Yang 2009, García and Strobl 2011, and Goldstein and Yang 2015). All of these papers explore information multiplicity in static or finite-horizon frameworks. In contrast, this is the first paper that examines a fully dynamic information market. Closely related are a set of papers that discusses similar sources of multiplicity in finite-horizon models (see Froot et al. (1992) and Avdis (2016)). In particular, Avdis (2016) proves that multiple equilibria can arise in a static information market followed by a dynamic financial market. Different from Avdis (2016) and that literature in general, this paper’s result and analysis refer to steady states and dynamic multiplicity, rather than only equilibrium multiplicity in the static sense. The perpetual dynamic framework here allows for characterization of the evolution and transition paths of information choices in response to time-varying uncertainty. Mele and Sangiorgi (2015) also explore market reactions to changes in uncertainty in a static model in which investors are subject to Knightian uncertainty. They do not explore how supply shocks affect the effectiveness
of information markets in accommodating uncertainty shocks or the nature of coordination failure in a dynamic information market.

The paper is also related to the literature that studies infinite-horizon trading models with asymmetric information, pioneered by Wang (1993, 1994) and Campbell and Kyle (1993). It is particularly related to models with overlapping generations of investors (Spiegel, 1998; Bacchetta and Van Wincoop, 2006; Watanabe, 2008; Biais et al., 2010; Albagli, 2015). All of these papers take information acquisition as exogenous. Also, these papers only consider first-moments shocks and focus on steady states where pricing coefficients are time-invariant constants. This paper explores time-varying second-moments and dynamic information choice. It develops a methodology to solve for nonstationary equilibria where pricing coefficients are allowed to vary systematically as functions of the state variables in the presence of steady state multiplicity.

A more recent literature explores the implications of dynamic information acquisition on long-run trends observed in (international) financial markets (Farboodi and Veldkamp 2017, Farboodi et al. 2018, and Valchev 2017). Brunnermeier et al. (2017) illustrates that active government intervention may harm information efficiency by leading investors to learn about noises introduced through the intervention. Although similar dynamic coordination forces exist in their frameworks, they do not address the issue of multiplicity; nor do they focus on the role of information markets in countering uncertainty shocks.

2 Model Economy

The model can be viewed as a perpetual repetition of Grossman and Stiglitz (1980) static model with long-lived assets. The physical environment is identical to Wang (1994), except for the overlapping-generations structure and endogenous information market. Time is discrete and runs from $-\infty$ to $+\infty$. The economy is populated by a continuum of overlapping-generations agents who consume a single consumption good. The good is treated as the numeraire. There are two

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2Dow and Gorton (1994) study a dynamic overlapping-generations model with private information in which, similar to this paper, a dynamic informational linkage is present: information gets incorporated into the price only if informed traders expect future traders to also impound their information in the price. Unlike this paper, however, it does not address the issue of multiplicity. I thank a referee for pointing this out.

3For a discussion of the relations these two ingredients, see Section 6.
assets in the economy: a bond in perfect elastic supply, paying a return \( R \), and a stock that pays a dividend

\[
D_t = F_t + \varepsilon_t^D
\]  

(2.1)
each period. \( F_t \) is the persistent component of the dividend process, labeled as the stock fundamental. The stock fundamental follows an AR(1) process:

\[
F_t = \rho^F F_{t-1} + \varepsilon_t^F, \quad 0 \leq \rho^F \leq 1.5
\]  

(2.2)
The stock supply, \( x_t \), follows an AR(1) process as well:

\[
x_t = \rho^x x_{t-1} + \varepsilon_t^x, \quad 0 \leq \rho^x \leq 1.
\]  

(2.3)

As in Wang (1994), I assume that there is a public signal every period about the current fundamental:

\[
S_t = F_t + \varepsilon_t^S.
\]  

(2.4)
The shock vector \( \varepsilon_t = [\varepsilon_t^D, \varepsilon_t^F, \varepsilon_t^x, \varepsilon_t^S] \) is i.i.d. over time, with mean 0 and covariance matrix \( \text{diag}(\sigma_D^2, \sigma_F^2, \sigma_x^2, \sigma_S^2) \).

Investors live for two periods with exponential utility.\(^6\) They are born with a certain amount of wealth \( w \) and freely observe the entire history of the dividend and stock price. They are then offered an opportunity to acquire information at some cost \( \chi \). If they choose to acquire information, they also observe the history of the stock fundamental. I call investors who choose to acquire information the “informed” investors and the rest “uninformed.” Denote the time \( t \) share of informed investors \( \lambda_t \) and the equilibrium stock price \( P_t \). The information set of the generation-\( t \) uninformed is

\[
\Omega^U_t = \{P_s, D_s, S_s\}_{s=-\infty}^t,
\]

and that for the informed is

\[
\Omega^I_t = \{P_s, D_s, S_s, F_s\}_{s=-\infty}^t.
\]

\(^4\)Alternatively, one can interpret the bond as a storage technology without nonnegative constraint.

\(^5\)The restriction that \( \rho^F > 0 \) is an economic one. Mathematically, we could allow \( \rho^F \) within the unit circle \((-1, 1)\).

\(^6\)An alternative model is that investors live forever but are myopic when making investment decisions. Given exponential utility, and thus no wealth effect, the two models are isomorphic.
An informed investor, observing the true stock fundamental, can perfectly infer the stock supply. For uninformed investors, their conditional expectations are derived from Kalman filter equations. We use \( \hat{F} \) and \( \hat{x} \) to denote the conditional mean of the current fundamental and stock supply for the uninformed:

\[
\begin{align*}
\hat{F}_t &= E(F_t | \Omega^U_t) \quad (2.5) \\
\hat{x}_t &= E(x_t | \Omega^U_t). \quad (2.6)
\end{align*}
\]

After the information acquisition stage, the financial market opens and trade occurs. After that, old investors exit and consume their wealth. The timeline is summarized in Figure 1. The period-\( t \) investors’ problem is as follows. Upon birth, they make information acquisition decision

\[
\max \{ W^I_t, W^U_t \},
\]

where \( W^I_t \) denotes the expected utility of generation-\( t \) informed investors, and \( W^U_t \) denotes the expected utility for the generation-\( t \) uninformed. Then, conditional on the information set, they
make their portfolio choice to maximize expected utility derived from terminal consumption:

\[ W_i^t = \max_{s,c} E(-\exp(-\alpha c)|\Omega_t^i) \]
\[ c \leq (D_{t+1} + P_{t+1} - RP_t)s + R(w-1 \{ i = I \} \chi), \]  

(2.7)

where \( s \) denotes the number of stock shares to purchase, \( c \) denotes terminal consumption, and \( w \) denotes their initial wealth. \( \alpha \) is the risk-averse parameter.

It is challenging to solve noisy rational expectations models with general, potentially nonlinear, price functions. Breon-Drish (2015) shows that the linear equilibrium is the unique continuous equilibrium in the static model of Grossman and Stiglitz (1980). It therefore stands to reason that the dynamic model considered here has the same linear equilibrium as the unique continuous solution. I therefore focus on linear equilibria in which equilibrium stock price depends linearly on the (expected) stock fundamental and supply. That is, there exists a set of time-varying coefficients \( \{ \bar{p}_t, p_{F_t}, \hat{F}_t, p_{xt} \} \) such that

\[ P_t = \bar{p}_t + p_{F_t}\hat{F}_t + p_{F_t}F_t - p_{xt}x_t. \]  

(2.8)

Observing the stock price \( P_t \) is equivalent to observing a price signal:

\[ S_{pt} = F_t - \frac{p_{xt}}{p_{Ft}}x_t. \]

The price signal partially reveals the true value of stock fundamental \( F_t \). The informativeness of the signal depends positively on the endogenous informative ratio \( \frac{p_{Ft}}{p_{xt}} \). This ratio is an important determinant of investors’ conditional beliefs. I denote it \( \theta_t = \frac{p_{Ft}}{p_{xt}} \).

3 Characterization

The equilibrium is characterized by three dynamic relations between investors’ beliefs, pricing coefficients, and the share of investors acquiring information. The first is a Kalman filter equation that characterizes the evolution of the conditional expectations of uninformed investors. The second is a financial market clearing condition that pins down pricing coefficients. The third is an information optimality condition to guarantee optimality of investors’ information acquisition decision, which pins down the share of informed investors.
3.1 Conditional Expectations of uninformed investors

Uninformed investors form their beliefs based on the entire history of dividends, public signals, and equilibrium stock prices. I here argue that this history can be summarized in two state variables. One is prior fundamental uncertainty.

\[ z_{t-1} = Var(F_{t-1}|\Omega_{t-1}^U) \]

The other is the informative ratio of the last period:

\[ \theta_{t-1} = \frac{p_{F,t-1}}{p_{x,t-1}} \]

Both are necessary, because they affect the conditional joint distribution of the current fundamental and current price signal. To see this, I arrange the current price signal \( S_{pt} \) into a combination of last period fundamental, price signal and current period noise:

**Lemma 3.1**

\[ S_{pt} = \left( \rho^F - \rho^x \frac{\theta_{t-1}}{\theta_t} \right) F_{t-1} + \rho^x \frac{\theta_{t-1}}{\theta_t} S_{pt-1} + \varepsilon_F - \frac{1}{\theta_t} \varepsilon_x \] (3.1)

Thus, conditional on observing last period’s price signal, a greater \( \theta_{t-1} \) implies that current price signal \( S_{pt} \) is less responsive to last period’s stock fundamental. This reduces the conditional correlation between the current fundamental and price signal. To see this:

\[ Cov(F_t, S_{pt}|\Omega_{t-1}^U) = Cov(\rho^F F_{t-1} + \varepsilon_F, S_{pt}|\Omega_{t-1}^U) \]
\[ = \rho^F \left( \rho^F - \rho^x \frac{\theta_{t-1}}{\theta_t} \right) Var(F_{t-1}|\Omega_{t-1}^U) + \sigma_F^2 \] (3.3)

The next proposition characterizes how the conditional belief evolves:

**Proposition 3.1** Given the sequence of \( \{\theta_t\} \), the law of motion for conditional volatility \( z_t \) is

\[ \frac{1}{z_t} = \frac{1}{Var(F_t|\{P_t\} \cup \Omega_{t-1}^U)} + \frac{1}{\sigma_D^2} + \frac{1}{\sigma_S^2}, \] (3.4)

where \( Var(F_t|\{P_t\} \cup \Omega_{t-1}^U) \) is the fundamental volatility upon observing past history and the current

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\(^7\)In principle, the (expected) level of stock fundamental and stock supply are also part of the state. With a utility function that exhibits no wealth effect, however, pricing coefficients are determined only by the conditional volatility.
price signal (but not the dividend and public signal), and is a function of \(z_t\) and pricing coefficients \(p_{Ft}, p_{xt}\):

\[
\text{Var}(F_t|\{P_t\} \cup \Omega_{t-1}^U) = (\rho_F^2)^2 z_{t-1} + \frac{\sigma_F^2}{2} \left[ \rho_F^2 \left( \frac{\rho_F^x}{\sigma_x} \right) z_{t-1} + \frac{\sigma_F^2}{2} \right] \left( \frac{1}{\sigma_x^2} \right) \rho_F^x + \frac{1}{\sigma_x^2},
\]

(3.5)

Detailed proofs are relegated to the appendix. Observing that \(z_t = \text{Var}(F_t|\Omega_{t-1}^U)\), equation 3.4 follows immediately from the standard Kalman filter formula for normal variables. Equation 3.5 can be understood as follows. First, observing an additional price signal reduces fundamental uncertainty

\[
\text{Var}(F_t|\{P_t\} \cup \Omega_{t-1}^U) < \text{Var}(F_t|\Omega_{t-1}^U) = (\rho_F^2)^2 z_{t-1} + \frac{\sigma_F^2}{2}
\]

unless the current stock price is not uninformative \((\theta_t \to 0)\). Second, when stock supply \(x_t\) is i.i.d. \((\rho_x^2 = 0)\), the price signal is just the sum of true fundamental \(F_t\) and some noise term uncorrelated with past history. Thus, equation 3.4 collapses to the standard formula, whereby ex post precision is equal to the sum of ex ante precision and the precision of the price signal.

\[
\frac{1}{z_t} = \frac{1}{(\rho_F^2)^2 z_{t-1} + \frac{\sigma_F^2}{2}} + \frac{1}{\frac{1}{\sigma_x^2}} + \frac{1}{\frac{1}{\sigma_{D}^2}} + \frac{1}{\frac{1}{\sigma_S^2}},
\]

(3.6)

Note that \(\theta_{t-1}\) drops out of the expression. This implies that the state space collapses to a single dimension of \(z_{t-1}\). When stock supply is persistent, \(x_t\) is correlated with past history, and thus one needs to invoke the projection theorem of normally distributed variables to obtain equation 3.5. This suggests that in general, we need to keep track of both \(z_t\) and \(\theta_t\) when solving for model dynamics.

Next we consider the evolution of conditional mean \(\hat{F}_t\). This is useful when we get to the characterization of optimal portfolio choices.

**Lemma 3.2** Given the sequence of pricing coefficients \(\{\theta_t, p_{xt}\}\) and state variables \(\{z_t\}\), the law of motion for the conditional mean \(\hat{F}_t\) is given by:

\[
\hat{F}_{t+1} = f_{\hat{F}_{t+1}} \hat{F}_t + f_{Ft+1} F_t + \tilde{f}_{\hat{F}t+1} \tilde{E}_{t+1},
\]

(3.7)

where \(f_{\hat{F}_{t+1}}\) and \(f_{Ft+1}\) are scalars and \(\tilde{f}_{\hat{F}t+1}\) is a 1 by 4 vector of coefficients. These coefficients
depend on \(\theta_t, z_t, \theta_{t+1}, z_{t+1}\). \(\bar{\varepsilon}_{t+1} = [\varepsilon_{t+1}^D, \varepsilon_{t+1}^F, \varepsilon_{t+1}^x, \varepsilon_{t+1}^S]\) is the 4 by 1 shock vector realized at time \(t+1\). In particular, its sensitivity with respect to \(F_t\) is given by

\[
f_{Ft+1} = \frac{\text{Var}(F_{t+1} | \Omega_{t+1})}{\text{Var}(F_{t+1} | \{P_{t+1}\} \cup \Omega_{t+1})} \left( \left( \rho^F - \rho^x \frac{\theta_t}{\theta_{t+1}} \right)^2 \text{Var}(F_t | \Omega_t^U) + \sigma_F^2 \right) \frac{\rho^F - \rho^x \frac{\theta_t}{\theta_{t+1}}}{\sigma_F^2} \text{Var}(F_t | \Omega_t^U) + \sigma_F^2 + \frac{1}{\theta_{t+1}^2} \sigma_x^2 \right)
\]

\[
\frac{\sigma_D^2}{\rho^F} + \frac{\rho^F \text{Var}(F_t | \Omega_t^U)}{\sigma_S^2} \rho^F
\]

That the law of motion for \(\hat{F}_t\) is linear is a standard property of the Kalman filter. Equation 3.8 measures how the current stock fundamental affects the future expected fundamental. A good realization of the current fundamental implies on average a favorable price signal, dividend signal, and public signal, all three of which imply a higher future expected fundamental. The three terms in the expression \(f_{Ft+1}\) capture the weight investors assign to each signal. These weights are determined by the relative precision of each signal and thus depend on price informativeness \(\theta\) and fundamental uncertainty \(z\).

### 3.2 Excess Stock Return and Optimal Portfolios

Given the equilibrium price function and uninformed investors’ beliefs, one can derive the expression for the excess stock return and optimal portfolios. The excess stock return consists of dividends and capital gains, less the interest cost of holding the stock:

\[
Q_{t+1} = D_{t+1} + P_{t+1} - RP_t
\]

\[
= F_{t+1} + \varepsilon_{t+1}^D + \bar{p}_{t+1} + p_{Ft+1} \hat{F}_{t+1} + p_{Ft+1} \hat{F}_{t+1} - p_{xt+1} x_{t+1} - RP_t
\]

where we substitute out stock dividend and equilibrium stock price. Using the law of motion for \(F_{t+1}\) (equation 2.2), \(x_{t+1}\) (equation 2.3), and \(\hat{F}_{t+1}\) (equation 3.7), \(Q_t\) can be expressed as a linear combination of time \(t\) variables and time \(t+1\) shocks.
Lemma 3.3 The excess stock return $Q_{t+1}$ can be expressed as

$$Q_{t+1} = \bar{q}_{t+1} + q_{ft+1} \tilde{F}_t + q_{ft+1} F_t - q_{xt+1} x_t + \bar{q}_{xt+1} \tilde{\varepsilon}_{t+1} - RP_t$$

where $q_{ft+1}, q_{Ft+1}, q_{xt+1} > 0$ are scalars and $\bar{q}_{xt+1}$ is a $1$ by $4$ vector. All coefficients may depend on $\theta_t, z_t, \theta_{t+1}, z_{t+1},$ and $p_{xt+1}$. In particular:

$$q_{Ft+1} = \rho^F (1 + p_{Ft+1}) + p_{Ft+1} f_{Ft+1}$$
$$q_{xt+1} = \rho^x p_{xt+1}.$$  

$q_{Ft+1}$ is the sensitivity of the future stock return with respect to the current stock fundamental (equation 3.11). The first term $\rho^F (1 + p_{Ft+1})$ captures the direct favorable impact of current fundamental on future dividends and stock prices due to its persistence. The second term $p_{Ft+1} f_{Ft+1}$ captures a signaling effect whereby innovations in the stock fundamental impacts uninformed investors’ belief through more favorable signals. Equation 3.12 indicates that persistent supply shocks have a negative impact on the future stock return, as it predicts unfavorable future stock supply. Note that knowing the informative ratio is not sufficient to characterize the excess stock return; one also needs information of $p_{xt+1}$.

Lemma 3.3 suggests that the excess stock return can be decomposed into the following three components in terms of information content:

$$Q_{t+1} = \underbrace{\bar{q}_{t+1}}_{\text{known to all}} + \underbrace{q_{ft+1} \tilde{F}_t}_{\text{known to informed only}} - \underbrace{RP_t}_{\text{known to all}} + \underbrace{q_{ft+1} F_t}_{\text{known to informed only}} - \underbrace{q_{xt+1} x_t}_{\text{known to informed only}} + \underbrace{\bar{q}_{xt+1} \tilde{\varepsilon}_{t+1}}_{\text{not known to either}}$$

The first component consists of a constant, current stock prices, and uninformed investors’ belief $\tilde{F}_t$. These are known to all agents in the economy. The second component consists of the realized current stock fundamental and stock supply. This information is known only to the informed investors. The third component consists of future noises that no one at period $t$ could possibly know. Thus the conditional volatility of the excess stock return for informed investors is just the volatility of the noise term:

$$V^I_t = Var(Q_{t+1} | \Omega^I_t) = Var(\tilde{q}_{xt+1} \tilde{\varepsilon}_{t+1} | \Omega^I_t)$$

(3.14)
Since the vector of shock coefficients depends on $p_{F_{t+1}}, p_{F_{t+1}}$ and $z_{t+1}$, $V_t^I$ also depend on these variables. The conditional volatility faced by the uninformed also reflects the fact that current investors are uncertain about the current stock fundamental and stock supply:

$$V_t^U = Var(Q_{t+1}|\Omega^U) = Var(q_{F_{t+1}}F_t - q_{xt+1}x_t + \vec{q}_{xt+1}\vec{\varepsilon}_{t+1}|\Omega^U)$$

$$= Var(q_{F_{t+1}}F_t - q_{xt+1}x_t|\Omega^U) + V_t^I. \quad (3.15)$$

Note that uninformed investors observe the current price signal. Using the price signal to substitute out supply $x_t$,

$$V_t^U = Var(q_{F_{t+1}}F_t - q_{xt+1}\frac{p_{F_{t+1}}}{p_{xt}}(F_t - S_{pt})|\Omega^U) + V_t^I$$

$$= Var(q_{F_{t+1}}F_t - q_{xt+1}\frac{p_{F_{t+1}}}{p_{xt}}\Omega^U_t) + V_t^I$$

$$= \left(q_{F_{t+1}} - q_{xt+1}\frac{p_{F_{t+1}}}{p_{xt}}\right)^2 Var(F_t|\Omega^U_t) + V_t^I$$

$$= (q_{F_{t+1}} - q_{xt+1}\theta_t)^2 z_t + V_t^I \quad (3.16)$$

Given the expected stock returns and conditional volatility, we can now derive investors’ optimal portfolio. As investors live for two periods with exponential utility, their optimal portfolio choice is particularly simple:

$$s_t^i = \frac{E(Q_{t+1}|\Omega_t^i)}{\alpha V_t^i}, i = I, U$$

Denoting the time $t$ share of informed investors $\lambda_t$, the pricing coefficients $p_{Ft}$ and $p_{xt}$ are then determined by matching coefficients so that the market clearing condition holds for any realizations of the stock fundamental and supply. These conditions implicitly define the law of motion for $\theta_t$ and $p_{xt}$:

**Proposition 3.2** Given a sequence of conditional volatility $\{z_t\}$ and share of informed investors $\{\lambda_t\}$, the following two equations define the law of motion for pricing coefficients $\theta_t$ and $p_{xt}$:

$$\left[\lambda_t \frac{1}{\alpha V_t^I} + (1 - \lambda_t) \frac{1}{\alpha V_t^U}\right] (R p_{xt} - \rho^x p_{xt+1}) = 1 \quad (3.18)$$

$$\lambda_t \frac{q_{F_{t+1}} - \rho^x p_{xt+1} \theta_t}{V_t^I} = \theta_t \quad (3.19)$$
where $V_t^I$ and $V_t^U$ are the conditional volatility of the stock return for informed and uninformed investors defined by equations 3.14 and 3.15 and are functions of $\theta_t$, $z_t$, $\theta_{t+1}$, $z_{t+1}$, and $p_{xt+1}$.

Equations 3.18 and 3.19 are forward-looking in nature. Investors make forecasts about future pricing coefficients and conditional distributions of the excess stock return, and their demand through market clearing delivers the current pricing coefficients. It is immediate to see that when there are no informed investors $\lambda_t = 0$, the informative ratio is equal to zero (equation 3.19).

### 3.3 Value of Information

We define the value of information as the ratio of the expected utility of the informed and uninformed investors.

**Definition 3.1** Define the value of information at time $t$:

$$I_t = W_t^U / W_t^I,$$

where the expected utilities $W_t^i$ are given by 2.7.

An elegant theoretical result of Grossman and Stiglitz (1980) is that the value of information only depends on second moments and is proportional to the ratio of the conditional volatility of excess stock return. This result carries over to this dynamic model because agents live for two periods, just as in Grossman and Stiglitz (1980).\(^8\) Thus conditional on the distribution of the excess stock return, they solve exactly the same problem as in Grossman and Stiglitz (1980) world, yielding the same expression of expected utility, and hence the same expression for the value of information.

**Proposition 3.3** The value of information is proportional to the ratio of conditional volatility of the excess stock return and is given by

$$I_t = e^{-\alpha R\chi} \sqrt{\frac{V_t^U}{V_t^I}},$$

where $V_t^I$ and $V_t^U$ are the conditional volatility of the stock return for informed and uninformed investors defined by equations 3.14 and 3.15 and are functions of $\theta_t$, $z_t$, $\theta_{t+1}$, $z_{t+1}$, and $p_{xt+1}$.

---

\(^8\)When agents’ horizon extends to more than two periods, this result no longer holds. In particular, the value of information would presumably depend on first moments of the model.
Information optimality means that investors have no incentive to deviate from their information status. In view of Proposition 3.3 and the fact that conditional volatilities are functions of \( \theta_t, z_t, \theta_{t+1}, z_{t+1}, \) and \( p_{xt+1} \), we can write the information optimality condition as follows:

\[
\mathcal{I}(\theta_t, z_t, \theta_{t+1}, z_{t+1}, p_{xt+1}) = 1 \quad \text{if} \quad \lambda_t \in (0, 1) \quad (3.20)
\]

\[
\mathcal{I}(\theta_t, z_t, \theta_{t+1}, z_{t+1}, p_{xt+1}) \leq (\geq) 1 \quad \text{if} \quad \lambda_t = 0(1)
\]

### 3.4 Summary

To summarize, the equilibrium is fully characterized by three dynamic relations between conditional volatility, pricing coefficients, and share of informed investors. The first relation is a Kalman filter equation that characterizes the evolution of conditional volatility (Proposition 3.1):

\[
\mathcal{K}(z_{t-1}, \theta_{t-1}, z_t, \theta_t) = 0 \quad (3.21)
\]

The second relation is a set of market clearing conditions that pin down the evolution of the pricing coefficients and thus informative ratio (Proposition 3.2):

\[
\mathcal{M}(\lambda_t, \theta_t, z_t, \theta_{t+1}, z_{t+1}, p_{xt+1}) = 0 \quad (3.22)
\]

The third relation is an information optimality condition that is given by equation 3.20.

The equilibrium is both backward-looking and forward-looking. On the one hand, the current market equilibrium depends on the entire history of information, summarized by the state variables \( z_{t-1} \) and \( \theta_{t-1} \). On the other hand, trading and information acquisition choice depend on future stock returns, which in turn depend on future pricing coefficients \( \theta_{t+1}, p_{xt+1} \) and future fundamental uncertainty \( z_{t+1} \). The next section starts by characterizing the steady-state property of the system.

### 4 Steady-state Multiplicity

This section analyzes the steady states of the model. The target is to uncover a key force in the dynamic information market: the dynamic complementarity in information acquisition whereby the
value of information today is increasing in the share of informed investors in the future. This has
two implications: First, multiple steady states could arise for appropriate levels of information cost.
With steady-state multiplicity, the variation of shooting method cannot be applied in solving for
model dynamics. This motivates the recursive method introduced in the following section. Second,
this dynamic complementarity effect is also the source of endogenous uncertainty shocks discussed
later.

I define the steady-state value of information as a function of the share of informed investors.
Intuitively, we ask what the value of information is in a world in which the share of informed
investors is fixed and the financial market is in equilibrium.

**Definition 4.1** The steady-state value of information is defined as

\[ I(\theta_t(\lambda), z_t(\lambda), \theta_{t+1}(\lambda), z_{t+1}(\lambda), p_{xt+1}(\lambda)) \],

where all implicit functions are defined by \( K = 0 \) and \( M = 0 \) with stationarity imposed. \( \lambda \) denotes
the steady-state share of informed investors and is exogenously fixed.

The strategy is to show that the steady-state value of information can increase with the steady-
state share of informed investors. As the steady-state share of informed investors varies, it affects
both today’s and future stock price. The future stock price then feeds back into today’s information
choice. This introduces an important feedback channel that is absent in Grossman and Stiglitz
(1980) where future stock payoff is an exogenous function.

To gain intuition, I focus on the information gain component of the value of information defined
as the amount of uncertainty reduced through information acquisition:

\[ \Delta V_t = V_t^U - V_t^I \] (4.1)

Using equation 3.17 and substituting in expressions \( q_{Ft+1} \) and \( q_{xt+1} \) given by lemma 3.3, we arrive
at the following expression:

\[ \Delta V_t = \left( \rho F(1 + p_{Ft+1}) + p_{Ft+1} f_{Ft+1} - \rho_x p_{xt+1} \frac{p_{Ft}}{p_{xt}} \right)^2 \text{Var}(F_t | \Omega_t^U) \] (4.2)
The information gain hinges upon two aspects: First, how much fundamental uncertainty faced by investors — the term $\text{Var}(F_t|\Omega^U_t)$ — and second, how sensitive the future stock return is with respect to current fundamental. When there is higher fundamental uncertainty or when return is more sensitive to the fundamental, there is greater gain from acquiring information. The endogenous sensitivity term is what distinguishes this paper from Grossman and Stiglitz (1980) in which sensitivity only comes from exogenous dividends. Here the sensitivity is endogenous, because the return also depends on the future resale stock price which in turn determined by future investors’ information acquisition and trading activities. This introduces a dynamic coordination motive within the dynamic information market.

As the share of steady-state informed investors increases, two opposing effects emerge. On the one hand, there are more informed investors today. This implies that the current price signal becomes more precise and thus there is less fundamental uncertainty. This is the classic static substitutability effect in Grossman and Stiglitz (1980). On the other hand, there are more informed investors in the future. Thus, future stock prices become more sensitive to fundamental: $p_{Ft+1}$ increases. This tends to increase the endogenous sensitivity component in the information gain. This dynamic complementarity force raises the value of information. Whether the value of information is upward-sloping generally depends on the relative strength of these two forces.

A crucial observation of this paper is that the static substitutability effect is locally absent around $\lambda = 0$. That is, $\text{Var}(F_t|\Omega^U_t)$ does not vary with $\lambda$ as it tends to zero. This observation follows from two properties of the Kalman filter equation $K$ (see proposition 3.1): First, $\lambda$ does not enter into the equation directly, but only indirectly through the pricing coefficients. Second, pricing coefficients $p_F$ and $p_x$ enter the equation only through the informative ratio squared $\theta^2$. Thus the derivatives always contain this ratio $\theta$, which converges to 0 when $\lambda$ tends to 0. These two properties, taken together, imply that the conditional volatility of the stock fundamental is not affected by changes in the share of informed investors, directly or indirectly. We can thus treat it as a constant when $\lambda$ is very close to 0.

We summarize this observation in the following proposition

**Proposition 4.1** (Local absence of static substitutability) Denote $\lambda$ the steady-state share of informed investors. The total derivative of conditional volatility with respect to the share of informed
investors, when \( \lambda \) tends to 0, converges to zero:

\[
\frac{d\text{Var}(F_t|\Omega^U_t)}{d\lambda} \to 0.
\]

Thus, when there are few informed investors, the dynamic complementarity effect always dominates. Rewrite equation 4.2 omitting the time script, and cancel out terms with respect to \( p_x \):

\[
\Delta V = (\rho^F(1 + p_F) + p_F f_F - \rho_x p_F)^2 \text{Var}(F|\Omega^U)
\]

A standard property of dynamic noisy rational expectation models is that \( p_F \) and \( p_F \) sum to a constant (Wang, 1994). Denoting this constant \( a \), we have:

\[
\Delta V = (\rho^F(1 + p_F) + (a - p_F)f_F - \rho_x p_F)^2 \text{Var}(F|\Omega^U)
\]

The coefficient \( f_F \) depends on various volatility terms (see equation 3.8), which are also just functions of \( \theta^2 \). Thus, by the similar logic of Proposition 4.1, we know that the sensitivity parameter \( f_F \) does not vary with \( \lambda \) when \( \lambda \) is near zero. Thus, \( \lambda \) can affect information gain \( \Delta V \) only through pricing coefficient \( p_F \). Taking this derivative yields the following expression:

\[
\frac{d\Delta V}{d\lambda} = 2 (\rho^F(1 + p_F) + (a - p_F)f_F - \rho_x p_F) \left( \rho^F - f_F - \rho_x \right) \frac{dp_F}{d\lambda}
\]

where in the second line I substitute in the fact that \( p_F \) tends to zero when \( \lambda \) tends to zero. To the extent that increasing \( \lambda \) naturally increases \( p_F \), the sign of this derivative hinges critically on the term

\[
\rho^F - f_F - \rho_x
\]

When there are no informed investors, market learning does not arise. Thus, uninformed investors’ beliefs only depend on the precision of dividend and public signals. Thus the sensitivity parameter \( f_F \) collapses to:

\[
f_F = \rho^F \left( \frac{\text{Var}(F|\Omega^U)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega^U)}{\sigma_S^2} \right)
\]

These steps deliver the following proposition characterizing the slope of information gain:
Proposition 4.2

\[ \frac{d\Delta V}{d\lambda} > 0 \text{ for } \lambda \text{ sufficiently small} \]

if and only if

\[ \left( 1 - \frac{\text{Var}(F|\Omega^U)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega^U)}{\sigma_S^2} \right) \rho^F - \rho^x > 0 \]

where \( \text{Var}(F|\Omega^U) \) is the steady-state conditional volatility of the stock fundamental given by Proposition 3.1 with \( \lambda = 0 \).

The proposition reveals that the dynamic complementarity force generally depends on three aspects. First, a more persistent stock fundamental strengthens dynamic complementarity, as the future stock return would be more sensitive to the current fundamental; second, a more persistent stock supply weakens dynamic complementarity. This is because with persistent supply, information about the stock fundamental is not that useful for predicting the future stock return in addition to the current price signal. The logic is that conditional on current price signal, observing a good fundamental also implies a large stock supply. If this large supply persists into the future, the future stock return would not be very high despite the favorable fundamental information. Thus, more persistent stock supply reduces the sensitivity of the future stock return with respect to the fundamental, making information acquisition less attractive.\(^9\) Third, it also depends on the precision of exogenous signals, as this affects how aggressively uninformed investors trade. When the exogenous signals are very precise, varying the share of informed investors would not have a substantial effect on the loading coefficients of the stock fundamental as both types of investors trade equally aggressively on fundamental information. This reduces the strength of dynamic complementarity.

In general, the value of information \( I \) not only depends on the information gain \( \Delta V \) but also on the level of conditional volatility faced by the informed investors \( V^I \):

\[ I = e^{-\alpha R_x} \sqrt{ \frac{V^U}{V^I} } = e^{-\alpha R_x} \sqrt{ 1 + \frac{\Delta V}{V^I} } . \]

This introduces an additional term in the complete characterization of the value of information, as stated by the following theorem:

\(^9\)A version of this effect also appears in the finite-horizon model of Avdis (2016).
Theorem 1 Fix a financial market equilibrium \( j = h, l \):

\[
\frac{dI}{d\lambda} > 0 \text{ for } \lambda \text{ sufficiently small.}
\]

if and only if

\[
\left(1 - \frac{\text{Var}(F|\Omega^U)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega^U)}{\sigma_S^2}\right) \rho^F - \rho^p + \phi^j > 0
\]

where \( \text{Var}(F|\Omega^U) \) is the steady-state conditional volatility of the stock fundamental given by Proposition 3.1 taking \( \lambda = 0 \). Parameter \( \phi^j \) depends on the type of financial market equilibrium \( j = h, l \).

The additional term \( \phi^j \) generally depends on the type of financial market equilibrium. This is because the volatility introduced by noise trading enters into the uncertainty faced by informed investors \( V^I \). This leads to interesting interactions between the financial market equilibrium and information market equilibrium. As the focus of this paper is on dynamics, interested readers are referred to the previous version of this paper for results in this regard.

5 Dynamics with Time-varying Uncertainty

5.1 Recursive Linear Equilibrium (RLE)

Going beyond steady state requires a method to solve for time-varying pricing coefficients. One possibility is to use the variation of shooting method whereby one fixes a steady state in the long run and works out the transition path by backward induction. This approach does not work with multiple steady states in the information market, because one is not certain which steady state the economy would converge to.

In view of this difficulty, I use an alternative recursive method to solve for model dynamics. Specifically, I look for equilibria in which pricing coefficients are time-invariant functions of the state variables \((z_{t-1}, \theta_{t-1})\). This notion of equilibrium allows for time-varying pricing coefficients in a systematic way, governed by predetermined state variables. The state variables summarize all past histories and information, and is the only payoff-relevant state of this system.\(^{10}\) Another advantage of this approach is a clear exposition of the respective roles played by each state variables

\(^{10}\)That equilibrium prices can only depend on the payoff-relevant state rules out the possibility of reputation concerns. In the overlapping-generation framework with no bequest motive, however, this concern is likely irrelevant.
(z vs. \(\theta\)) in determining the response of the economy to uncertainty shocks.

**Definition 5.1** A recursive linear equilibrium (RLE) consists of a set of functions \(z'(z, \theta), \theta'(z, \theta), \) \(p_x(z, \theta), \) and \(\lambda(z, \theta)\) such that the time series \(\{z_t, \theta_t, p_{xt}, \lambda_t\}\) generated by these functions \(z_t = z'(z_{t-1}, \theta_{t-1}), \theta_t = \theta'(z_{t-1}, \theta_{t-1}), \) \(p_{xt} = p_x(z_{t-1}, \theta_{t-1}), \) and \(\lambda_t = \lambda(z_{t-1}, \theta_{t-1})\) satisfies the Kalman filter equation 3.21, the market clearing conditions 3.22, and the information optimality condition 3.20.

### 5.2 A One-shot Theorem

It is generally hard, if not impossible, to analytically characterize RLE, particularly when the dynamic system is multidimensional. However, a lot can be said in the special case in which the stock supply stock is i.i.d. over time: \(\rho^x = 0.\) As stock supply becomes a pure noise, \(\theta_{t-1}\) no longer affects the conditional distribution between fundamental and price signal. Thus it drops out of the evolution of investors’ conditional beliefs, and conditional volatility \(z_t\) becomes the only state variable (see equation 3.6).

Given this simplification, we are able to prove the following theoretical result:

**Theorem 2 (One-shot Theorem)** Suppose \(\rho^x = 0.\) There exists an RLE satisfying the following property: for any \(z_{t-1},\)

\[
z'(z_{t-1}) = \tilde{z}^{ss} \quad \text{whenever} \quad \lambda(z_{t-1}) \in (0, 1)
\]

where \(\tilde{z}^{ss}\) is the steady-state level of conditional volatility associated with the RLE. Moreover:

\[
\lambda'(z_{t-1}) \geq 0 \\
\theta'(z_{t-1}) \geq 0
\]

Imagine that the economy has been operating at a steady state up to period \(t-1.\) At the end of period \(t-1\) there is an unexpected uncertainty shock that raises the conditional uncertainty faced by uninformed investors. The value of information increases, and the information market responds by increasing the share of informed investors (\(\lambda'(z_{t-1}) \geq 0\)). This delivers a more precise price
signal \((\theta'(z_{t-1}) \geq 0)\). The theorem states that, as long as the information market is operative, in the sense that it does not hit boundary \((\lambda(z_{t-1}) \in (0,1))\), such response would *exactly* offset the impact of uncertainty shock, so that the economy reverts to the steady state in the following period.

I will provide a constructive proof of this theorem. That is, for any starting value of \(z_{t-1}\) representing an unanticipated uncertainty shock, I will construct an RLE in which the evolution of \(z_t\) is given by \(z_{t+j} = z^{ss}, \forall j \geq 0\).

Given this constant future path of \(z_t\), the economy will be at the steady state starting from period \(t+1\), and thus all future equilibrium conditions are automatically satisfied. We therefore only need to verify period \(t\) equilibrium conditions. There are three equilibrium conditions — the Kalman filter equation, the financial market clearing condition, and the information optimality condition — but only two degrees of freedom: \(\theta_t\) and \(\lambda_t\). Since there is one less degree of freedom, generically not all equilibrium conditions can be met.

The key insight of the proof is that when \(\rho^x = 0\), the information optimality condition is redundant. The reason is that the value of information in this case does not depend directly on the current informative ratio \(\theta_t\); it only depends on future variables and current fundamental uncertainty \(z_t\), and is thus at its steady-state level when \(z_t\) is so. Hence the information market must be in equilibrium by the definition of steady state. To see this more precisely, focus on the conditional volatility of excess stock return \(V_t^I\) and \(V_t^U\). Equation 3.14 shows that the conditional volatility faced by informed investors \(V_t^I\) only depends on \(t+1\) variables, and thus must be at its steady state level. Equation 3.17 shows that the belief of uninformed investors is a function of \(V_t^I\) as well as \(z_t\) and \(\theta_t\):

\[
V_t^U = (q_{Ft+1} - q_{xt+1} \theta_t)^2 z_t + V_t^I
\]

Note that when \(\rho^x = 0\), supply becomes a pure noise and thus the future stock return is unaffected by current realization of stock supply. This implies that \(q_{xt+1} = 0\) (equation 3.12) and \(\theta_t\) drops out of the equation. Given that \(z_t\) is at its steady-state level, \(V_t^U\) is also at its steady-state level. As the value of information only depends on the conditional volatility of future stock return \(V_t^I\) and \(V_t^U\), it is also at its steady-state level. Thus, the information optimality condition is always satisfied, irrespective of what the values of \(\theta_t\) and \(\lambda_t\) are.
Now we have two variables \((\theta_t, \lambda_t)\) and two equilibrium conditions given by Kalman filter 3.6 and financial market clearing 3.19. Simple manipulations yield the following closed-form expressions for the two variables:

\[
\begin{align*}
\theta_t &= \sigma_x \sqrt{\frac{1}{z_t} - \frac{1}{(\rho_F^2) z_{t-1} + \sigma_F^2} - \frac{1}{\sigma_D^2} - \frac{1}{\sigma_S^2}} \\
\lambda_t &= \frac{\theta_t V_t^{t+1}}{q_F^{t+1}}
\end{align*}
\]

Note that \(z_t, V_t^t, q_{Ft+1}^t\) and \(q_{Ft+1}^t\) are all constants at steady-state values. This implies that both \(\theta_t\) and \(\lambda_t\) increases with \(z_{t-1}\).

So far our discussion implicitly assumes away the possibility that the information market hits boundary at time \(t\). When there is a particularly severe uncertainty shock, the information market may not be able to completely undo the shock as the upper bound on the number of informed investors is reached \((\lambda_t = 1)\). As a result, heightened uncertainty persists into the future. As analytical solutions are not available in this case, we now turn to numerical analysis.

### 5.3 Numerical Implementation

The model is parameterized in such a way that multiple steady states arise in the information market. Risk aversion parameter \(\alpha\) is set to 1. Risk free interest rate \(R\) is set to 1.05 so that a model period is one year.\(^{11}\) The volatility of supply innovation \(\sigma_x^2\) is set to 0.01, corresponding to an annual turnover rate of 10%. The volatility of fundamental innovation \(\sigma_F^2\) is set to 0.1. The volatility of both the stock dividend and public signal are set to be relatively large, \(\sigma_D^2 = \sigma_S^2 = 1\), so that there is significant information asymmetry in the financial market. Finally, I set a relatively high fundamental persistent \(\rho_F = 0.9\) and set supply persistence \(\rho_x\) to 0 as the benchmark. The information cost parameter \(\chi\) is set to 0.085 to guarantee that information multiplicity arises for all numerical exercises conducted in this paper. Note that this is not meant as a fully quantitative exercise, but rather an informed numerical exploration of model dynamics with information multiplicity.

Given the parameterization, there exist multiple financial market equilibria with different stock

\(^{11}\)Thus the implied investment horizon is also one year.
market volatility. Both types of financial market equilibria have nice properties, and this paper does not take a stand on which equilibria one should pick. I choose to focus on the low-volatility equilibrium to illustrate model dynamics, because it is stable, is the unique limit of an overlapping-generation model when trades become infinitely lived, and therefore is more comparable to classic papers such as Wang (1994).\textsuperscript{12}

The RLE can be solved using the standard time iteration method (see, for example, Coleman 1991). We start with an initial guess of the equilibrium functions and treat them as the equilibrium functions that generate time $t + 1$ variables and solve for time $t$ equilibrium functions. When solving for time $t$ equilibrium functions, we first fix the share of informed investors $\lambda_t$ and solve the pricing coefficients as functions of $\lambda_t$. We then measure the value of information and solve for the equilibrium $\lambda_t$ with the information optimality condition. We then update the equilibrium functions. We repeat this procedure until convergence. The detailed algorithm is available upon request.
5.4 Benchmark Results with $\rho^x = 0$

Figure 2 plots the steady state value of information as a function of share of informed investors. The value of information increases initially due to the dynamic complementarity effect and decreases as the share of informed investors approaches one. This is because the static substitutability effect is absent when there are few informed investors (see Proposition 4.1) and becomes prominent when the share of informed grows. Due to the nonlinear shape of the value of information, there are three steady states that can be ranked in terms of stock price informativeness. In the most informative steady state, 47% of investors are informed. The intermediate steady state has 9% of informed investors. At the least informative steady state, the value of information is strictly less than one and all investors remain uninformed.

The steady-state comparison provides natural candidates for initial guesses for pricing functions. In particular, there exist an upper bound and lower bound in terms of stock price informativeness. The upper bound comes from the steady state in which all investors are informed, and the least informative lower bound comes from the steady state in which none is informed.

I first examine the optimistic RLE obtained with the upper bound, i.e. the most informative initial guess. This corresponds to the case in which investors holds optimistic beliefs about future stock price informativeness. Results are presented in Figure 3. Panel A plots the law of motion for conditional volatility. Consistent with the one-shot theorem, I find that this function is constant: high uncertainty is exactly accommodated by the information market with increased share of informed investors (Panel B) and a more informative stock price (Panel C). More prior uncertainty reduces investors’ incentive to trade on their signals, therefore reducing the sensitivity of the stock price with respect to stock supply (Panel D).

To see the role of endogenous information in a more transparent way, I solve an alternative model in which the share of informed investors is exogenously fixed so that both the endogenous and the exogenous information model converge to the same steady state (i.e., the most informative steady state). I then compare the law of motion function $z'(.)$ from both models in Figure 4. The law of motion in the endogenous information model has a flatter slope, reflecting globally faster convergence to the steady state.

12 See the discussion in Albagli (2015). I thank a referee for pointing this out.
Figure 3: Equilibrium Functions of Optimistic RLE

Figure 4: Law of Motion for $z$ (Exogenous vs. Endogenous Information)
In Figure 5 I plot model impulse response to an unanticipated uncertainty shock. Suppose there is an exogenous shock to conditional volatility $z$ at time 1 such that the volatility is increased by 30%. The plots trace out the responses of uncertainty, share of informed investors, supply sensitivity, and stock price informativeness. Compared to its exogenous counterpart, the endogenous information model accommodates uncertainty shock by an increased share of informed investors (Panel B). This implies that the stock price becomes more sensitive to the fundamental and less sensitive to stock supply (Panel D), and hence becomes a more informative signal (Panel C). This effect is absent when information choice is exogenous; instead, supply sensitivity shoots up after the uncertainty shock (Panel D, blue curve), reflecting the fact that rational investors trade less aggressively on their information when facing higher uncertainty. This implies that supply shocks play a more important role in driving stock prices.

Endogenous Uncertainty Shocks and Dynamic Coordination Failure

So far we have analyzed the optimistic RLE based on the most informative initial guess. If we instead start with the least informative initial guess, one in which all investors are assumed to be uninformed, we would obtain a different RLE. I label this the pessimistic RLE, as it stems from investors’ pessimistic belief about future stock price informativeness. Figure 6 plots both RLEs along with a 45 degree line. The optimistic RLE converges to the most informative steady
state, whereas the pessimistic RLE converges to the least informative one in which no one chooses to become informed. The pessimistic RLE has a significantly steeper slope because, as shown in Figure 7, Panel A, the equilibrium share of informed investors is always zero, and thus the endogenous information channel is effectively shut down. It follows that in the pessimistic RLE, the stock price reveals no information about stock fundamental in addition to what uninformed investors already know (Panel C, figure 7).

The existence of multiple RLEs implies that the equilibrium is susceptible to changes in investors' belief. Figure 8 plots the transition path of the economy upon being hit by a pessimistic belief shock. The economy has been operating at the most informative steady state along the optimistic RLE up to time 0, and at time 1 there is a belief shock to the investors that shifts the economy to the pessimistic RLE. The share of informed investors instantly drops to zero (Panel B), along with stock price informativeness (Panel C). Uncertainty gradually grows as the economy converges to the least informative steady state (Panel A). Supply sensitivity displays an interesting nonlinear pattern (Panel D): It drops sharply at time 1 and gradually recovers a bit. The initial drop is due to a sudden rise in expected future uncertainty, causing rational uninformed investors to trade less aggressively on their noisy price signal. The later gradual recovery occurs because after the future belief stabilizes, the volume of trade from rational investors continues to shrink due to growing
uncertainty. Thus stock supply shocks become more important in driving stock prices.

The pessimistic RLE represents a situation of “dynamic coordination failure”. This arises because investors fail to coordinate across generations rather than within generations. This can be seen in Figure 9 in which I plot period $t$ value of information as a function of period $t$ share of informed investors, holding fixed future equilibrium functions and substituting in current market clearing conditions. I plot two cases in which the future equilibrium functions correspond to the optimistic and pessimistic RLE respectively. In both cases, I set prior uncertainty $z_{t-1}$ to 0.13. As illustrated in Figure 9, for both RLEs there is always a unique solution in the static information market, indicating that there is no static coordination issue. Multiplicity arises because of an intergenerational coordination failure: Period $t$ investors expect period $t + 1$ investors to not acquire information, who in turn expect period $t + 1$ investors to not acquire information. This dynamic complementarity is the source of multiple steady states, and is also the source of multiple RLEs here.

To the extent that more informative asset prices lower the cost of capital and are thus favorable from a policy perspective, regulators may have an incentive to regulate investors’ belief by making public announcements. The theory suggests that an effective announcement should emphasize what future investors would do rather than what investors are currently doing. The latter would not help
Figure 8: Optimistic vs. Pessimistic RLE

Figure 9: Static Value of Information
the situation, because there is no static coordination failure. This is in sharp contrast to predictions from finite-horizon models with static information multiplicity.

5.5 Persistent Stock Supply

I now turn to the case of persistent stock supply. When the stock supply is persistent, one must solve the dynamic system with two state variables: the prior uncertainty and the last period’s informative ratio. Figure 10 illustrates the equilibrium functions in this case. I plot various equilibrium functions with prior uncertainty $z_{t-1}$ on the $x$ axis holding fixed certain values of informative ratio $\theta_{t-1}$. Transition function $z'$ is upward-sloping, indicating that unlike the case in which $\rho^x = 0$, higher initial uncertainty could persist into the future. Interestingly, holding prior uncertainty fixed, a more precise price signal from the last period could amplify initial uncertainty. This can be seen in Panel A, in which the blue curve with high $\theta_{t-1}$ always lies above the red dashed curve with low $\theta_{t-1}$. The intuition is that, ceteris paribus, a more informative prior price signal reduces the conditional covariance between the current price signal and the fundamental:

$$\text{Cov}(F_t, S_{pt} | \Omega_{t-1}^U) \downarrow = \rho^F \left( \rho^F - \rho^x \frac{\theta_{t-1}}{\theta_t} \right) \text{Var}(F_{t-1} | \Omega_{t-1}^U) + \sigma_F^2.$$  

(5.2)

This makes the current price signal less useful in resolving fundamental uncertainty. As a result, high uncertainty persists into the future.

Overall there are two reasons why uncertainty shocks could have long-lasting impacts in this case. First, the persistent stock supply reduces the value of information, as the stock can be persistently underpriced. As a result, the future resale stock price can deviate significantly from the fundamental. This limits investors’ ability to exploit profitable arbitrage opportunities and thus renders information acquisition less attractive even when there is a large uncertainty shock. Second, a very informative price signal from the past reduces the effectiveness of the current price signal in resolving uncertainty. Thus, if an uncertainty shock hits an economy that was operating in a very informative environment, such a shock will be amplified by the high $\theta$ inherited from the past. In Figure 11, I compare the impulse response across two models with and without a persistent stock supply against an unanticipated uncertainty shock of 30%. One can see that uncertainty persists into the future in the model with a persistent stock supply (Panel A).
Interestingly, along the transition path following the uncertainty shock, we observe both a high quality of information and high level of fundamental uncertainty. This is because high prior uncertainty $z_{t-1}$ induces more investors to acquire information, leading to a high $\theta_t$. With a persistent stock supply, this high quality of stock price information is not sufficient to offset the impact of high prior uncertainty. Thus, $z_{t}$ is also higher than its steady-state level. This prediction does not arise when information choice is exogenous (blue line in figure 11). This is because higher uncertainty causes informed investors to trade less aggressively, leading to a less informative stock price (Panel C).

**Summary: Role of Supply Persistence**

So far we have shown that dynamic information acquisition acts as a double-edged sword in the financial system. On the one hand, it accommodates exogenous uncertainty shocks. On the other hand, the dynamic complementarity in information acquisition could lead to coordination failure, introducing endogenous uncertainty shocks to the system.

The key parameter that determines the strength of these two aspects is the persistence of stock supply. A more persistent stock supply weakens the dynamic complementarity (see Proposition 4.2 and Theorem 1), because it renders today’s value of information less responsive to the future share of

---

**Figure 10: Equilibrium Functions (Persistent Stock Supply)**

Panel A: $z'(z_{t-1})$

Panel B: $\lambda(z_{t-1})$

Panel C: $\theta'(z_{t-1})$

Panel D: $p(z_{t-1})$
informed investors. However, it also limits the ability of the information market in accommodating exogenous uncertainty shocks.

One could imagine that part of the supply shock comes from liquidity shocks to market participants. To the extent that the government can design subsidization policies to liquidity-constrained agents, it can affect the process of stock supply shock. This creates a policy tradeoff regarding the optimal degree of supply persistence. If investors’ belief is stable and there are frequent exogenous variations in uncertainty, it is optimal to make the supply shock less persistent. If large swings in investors’ beliefs are likely, it might be desirable to make the supply shock very persistent so that dynamic complementarity in the information market does not prevail.

6 Discussion

Identifying Exogenous vs. Endogenous Uncertainty Shocks

In the model, time-varying second moments of stock prices can be driven by both exogenous uncertainty shocks and endogenous information choices. Can we identify how much of the price variation is due to exogenous vs. endogenous sources? The model suggests an identification strategy
that measures stock price informativeness. Comparing Figure 5 and Figure 8 reveals that uncertainty shocks of different sources have very different implications on stock price informativeness. For uncertainty shocks of exogenous sources, endogenous information serves as a buffer against such shocks, with a greater share of investors acquiring information. As a result, stock price informativeness improves upon heightened uncertainty. For endogenous uncertainty shocks, heightened uncertainty is the outcome of a malfunctioning information market in which few investors acquire information. As a result, price informativeness and uncertainty negatively comove. Thus, different sources of uncertainty shocks predict different correlations between price informativeness and market uncertainty. This suggests that measuring the cyclical pattern of stock price informativeness could be helpful in understanding the relative importance of different sources of uncertainty shocks over the business cycle.

**Implications on Disclosure**

The flip side of an uncertainty shock is a public disclosure. Thus, our results on uncertainty shocks can also be used to think about effectiveness of information disclosure in the financial market. A classic result in the literature is that public disclosure tends to crowd out private information acquisition (see, for example, Verrecchia 1982 and Diamond 1985. For an excellent survey, see Goldstein and Yang, 2017). This paper adds to this literature by studying the dynamic effect of disclosure. The crucial parameter, in the dynamic model we consider here, is the persistence of stock supply shock. When the stock supply shock is i.i.d., the crowding out effect is so strong that the impact of disclosure can only last for one period. A more persistent stock supply implies that disclosure can have a more persistent impact on the economy.

**On Overlapping-generation Structure**

It is not possible to relax the assumption that agents are two-period lived within the Gaussian-linear framework. Recall that when investors are two-period lived, the value of information only depends on the second moments, which evolve deterministically. Once investors’ horizons are extended beyond two periods, the value of information also depends on the first moments of the model. This implies that the information choices and pricing coefficients become random variables, rendering the Gaussian-linear framework inapplicable.

For a related reason, we can only analyze unanticipated uncertainty shocks. If there are stochastic
second-moment shocks, pricing coefficients would become random variables and the normality struc-
ture of stock price would break down. Overall, exploring long-lived agents and expected uncertainty
shocks is interesting but seem to require a framework other than the traditional Gaussian-linear one.

7 Conclusion

This paper examines the interactions between dynamic information acquisition and time-varying
uncertainty in financial markets with asymmetric information. In particular, it explores the role of
dynamic information markets in accommodating exogenous uncertainty shocks, as well as creating
endogenous uncertainty shocks. It highlights the key role of stock supply persistence in determining
the strength of these two aspects: A more persistent stock supply makes endogenous uncertainty
shocks less likely, but also undermines the information market’s ability to accommodate exogenous
uncertainty shocks.

The theory can be extended in several dimensions. First, this paper only considers fundamental
analysis, as in Grossman and Stiglitz (1980). It would be interesting to incorporate supply analysis
(for example, Farboodi and Veldkamp 2017) into the framework and analyze how this affects time-
varying uncertainty. Second, this paper considers a single piece of fundamental information. In
reality, information can be diverse (Goldstein and Yang (2015)). It would be intriguing to explore
how the uncertainty of different dimensions of stock fundamental interact over time. Last, the
model features time-varying moments of stock prices driven by exogenous uncertainty shocks and
endogenous information choices. Thus, a fully quantitative extension of the model can be used
to understand the joint time-series pattern of the price volatility, uncertainty, and information
acquisition activities of, say, the US financial market. I leave these topics to future research.
References


Maryam Farboodi, Laura Veldkamp, and Adrien Matray. Where has all the big data gone. April 2018.


A Appendix

Proof of Lemma 3.1

\[ S_{pt} = F_t - \frac{p_{xt}}{p_{Ft}} x_t \]

\[ = F_t - \frac{p_{xt}}{p_{Ft}} (\sigma^F x_t + \epsilon_t^F) \]

\[ = F_t - \frac{p_{xt}}{p_{Ft}} (\rho^F p_{xt-1}) (F_{t-1} - S_{pt-1}) + \epsilon_t^F \]

\[ = F_t - \rho^F p_{xt} p_{Ft-1} (F_{t-1} - S_{pt-1}) - \frac{p_{xt}}{p_{Ft}} \epsilon_t^{x_t} \]

\[ = \rho^F F_{t-1} + \epsilon_t^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} (F_{t-1} - S_{pt-1}) - \frac{p_{xt}}{p_{Ft}} \epsilon_t^{x_t} + \rho^F \frac{p_{xt}}{p_{Ft}} S_{pt-1} \]

\[ = \left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right) F_{t-1} + \epsilon_t^F - \frac{p_{xt}}{p_{Ft}} \epsilon_t^{x_t} + \rho^F \frac{p_{xt}}{p_{Ft}} S_{pt-1} \]

Proof of Proposition 3.1

The price signal \( S_{pt} \) and stock fundamental \( F_t = \rho^F F_{t-1} + \epsilon_t^F \) are jointly normally distributed with mean:

\[ \begin{pmatrix}
\rho^F E(F_{t-1}|\Omega_{t-1}^U) \\
\left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right) E(F_{t-1}|\Omega_{t-1}^U) + \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} S_{pt-1}
\end{pmatrix} \]

and variance:

\[ \begin{pmatrix}
\left( \rho^F \right)^2 \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 \\
\left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right)^2 \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 \\
\left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right) \rho^F \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 \\
\left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right) \left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right)^2 \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 + \left( \frac{p_{xt}}{p_{Ft}} \right)^2 \sigma_x^2
\end{pmatrix} \]

By Projection Theorem

\[ \text{Var} \left( F_t | \{ P_t \} \cup \Omega_{t-1}^U \right) = \text{Var} \left( F_t | \Omega_{t-1}^U \right) - \frac{\left[ \text{Cov} \left( F_t, S_{pt} | \Omega_{t-1}^U \right) \right]^2}{\text{Var} \left( S_{pt} | \Omega_{t-1}^U \right)} \]

\[ = \left( \rho^F \right)^2 \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 - \frac{\left( \left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right) \rho^F \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 \right)^2}{\left( \rho^F - \rho^F \frac{p_{xt}}{p_{Ft}} p_{Ft-1} \right)^2 \text{Var}(F_{t-1}|\Omega_{t-1}^U) + \sigma_p^2 + \left( \frac{p_{xt}}{p_{Ft}} \right)^2 \sigma_x^2} \]

Thus

\[ \text{Var} \left( F_t | \{ P_t \} \cup \Omega_{t-1}^U \right) = \left( \rho^F \right)^2 z_{t-1} + \sigma_p^2 - \frac{\left( \left( \rho^F - \rho^F \frac{\theta_t}{\eta_t} \right) \rho^F z_{t-1} + \sigma_p^2 \right)^2}{\left( \rho^F - \rho^F \frac{\theta_t}{\eta_t} \right)^2 z_{t-1} + \sigma_p^2 + \left( \frac{1}{\eta_t} \right)^2 \sigma_x^2} \]

Where \( \theta_t = \frac{p_{xt}}{p_{Ft}} \) measures informativeness of stock price at time \( t \).

And by standard Kalman filter Formula we can obtain \( \text{Var} \left( F_t | \Omega_{t-1}^U \right) \).

Proof of Lemma 3.2
With similar derivation as in Proposition X, one can show that the conditional mean of stock fundamental is

\[
E\left(F_{t+1} \mid \{P_{t+1} \cup \Omega_t^U\}\right) = E\left(F_{t+1} \mid \Omega_t^U\right) + \frac{\text{Cov}\left(F_{t+1}, S_{pt+1} \mid \Omega_t^U\right)}{\text{Var}\left(F_{t+1}, S_{pt+1} \mid \Omega_t^U\right)} \left(S_{pt+1} - E\left(S_{pt+1} \mid \Omega_t^U\right)\right)
\]

\[
= \rho^F E\left(F_t \mid \Omega_t^U\right) + \left[\left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) \rho^F \text{Var}\left(F_t \mid \Omega_t^U\right) + \sigma_F^2\right]^2 \left(S_{pt+1} - \left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) E\left(F_t \mid \Omega_t^U\right) - \rho^s \frac{\theta_t}{\theta_{t+1}} S_{pt}\right)
\]

The posterior mean is a weighted average of ex-ante mean and signals:

\[
E\left(F_{t+1} \mid \Omega_t^U\right) = E\left(F_{t+1} \mid \{P_{t+1}, D_{t+1}, S_{t+1}\} \cup \Omega_t^U\right)
\]

\[
= \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\text{Var}\left(F_{t+1} \mid \{P_{t+1} \cup \Omega_t^U\}\right)} E\left(F_{t+1} \mid \{P_{t+1} \cup \Omega_t^U\}\right) + \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\sigma_D^2} D_{t+1} + \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\sigma_S^2} S_{t+1}
\]

Substituting in

\[
S_{pt+1} = \left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) F_t + \epsilon_{t+1}^F - \frac{1}{\theta_{t+1}} \epsilon_{t+1}^D + \rho^s \frac{\theta_t}{\theta_{t+1}} S_{pt}
\]

\[
D_{t+1} = F_{t+1} + \epsilon_{t+1}^D = \rho^F F_t + \epsilon_{t+1}^F + \epsilon_{t+1}^D
\]

\[
S_{t+1} = F_{t+1} + \epsilon_{t+1}^F = \rho^F F_t + \epsilon_{t+1}^F + \epsilon_{t+1}^D
\]

Collect terms, we obtain the law of motion for \(\hat{F}_t\):

\[
\hat{F}_{t+1} = f_{\hat{F}_{t+1}} \hat{F}_t + f_{F_{t+1}} F_t + f_{\tilde{f}_{t+1}} \tilde{f}_{t+1}
\]  

(A.1)

Where

\[
f_{\hat{F}_{t+1}} = \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\text{Var}\left(F_{t+1} \mid \{P_{t+1} \cup \Omega_t^U\}\right)} \left[\rho^F - \left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) \rho^F \text{Var}\left(F_t \mid \Omega_t^U\right) + \sigma_F^2\right]^2 \left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) \rho^F \text{Var}\left(F_t \mid \Omega_t^U\right) + \sigma_F^2 + \frac{1}{\sigma_{t+1}^2} \sigma_F^2\right]
\]

\[
f_{F_{t+1}} = \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\text{Var}\left(F_{t+1} \mid \{P_{t+1} \cup \Omega_t^U\}\right)} \left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) \rho^F \text{Var}\left(F_t \mid \Omega_t^U\right) + \sigma_F^2 + \frac{1}{\sigma_{t+1}^2} \sigma_F^2 \left(\rho^F - \rho^s \frac{\theta_t}{\theta_{t+1}}\right) + \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\sigma_D^2} \rho_F + \frac{\text{Var}\left(F_{t+1} \mid \Omega_t^U\right)}{\sigma_S^2} \rho_F
\]
The vector $\dot{f}_{t+1}$ consists of four entries:

$$f_1 = \frac{Var (F_{t+1}\mid \Omega_U^t)}{Var (F_{t+1}\mid \Omega_U^t)} \left[ \left( \rho^F - \rho^F \frac{\hat{q}_t}{\sigma^F} \right)^2 \frac{\rho^F Var (F_{t+1}\mid \Omega_U^t) + \sigma^F_2}{\sigma^F_D} \right] + Var (F_{t+1}\mid \Omega_U^t) \sigma^F_S$$

$$f_2 = -\frac{Var (F_{t+1}\mid \Omega_U^t)}{Var (F_{t+1}\mid \Omega_U^t)} \left[ \left( \rho^F - \rho^F \frac{\hat{q}_t}{\sigma^F} \right)^2 \frac{\rho^F Var (F_{t+1}\mid \Omega_U^t) + \sigma^F_2}{\sigma^F_D} \right] \frac{1}{\frac{\rho^F}{\sigma^F} - \theta^F_{t+1}}$$

$$f_3 = \frac{Var (F_{t+1}\mid \Omega_U^t)}{\sigma^F_D}$$

$$f_4 = \frac{Var (F_{t+1}\mid \Omega_U^t)}{\sigma^F_S}$$

**Proof of Proposition 3.2**

Given the expression for excess stock return $Q_{t+1}$, we can derive conditional mean for informed and uninformed.

For informed investors:

$$E \left( Q_{t+1}\mid \Omega_U^t \right) = \dot{q}_{t+1} + q_{Ft+1} \hat{F}_t + q_{Ft+1} F_t - q_{xt+1} x_t - RP_t$$

For uninformed investors:

$$E \left( Q_{t+1}\mid \Omega_U^t \right) = \dot{q}_{t+1} + q_{Ft+1} \hat{F}_t + q_{Ft+1} F_t - q_{xt+1} \hat{x}_t - RP_t$$

Use the relation

$$S_{pt} = F_t - \frac{1}{\theta^t} x_t = \dot{F}_t - \frac{1}{\theta^t} \hat{x}_t$$

to substitute out $\hat{x}_t$, we obtain

$$E \left( Q_{t+1}\mid \Omega_U^t \right) = \dot{q}_{t+1} + q_{Ft+1} \hat{F}_t + q_{Ft+1} \hat{F}_t - q_{xt+1} \left( \theta^t \left( \hat{F}_t - F_t \right) + x_t \right) - RP_t$$

$$= \dot{q}_{t+1} + \left( q_{Ft+1} + q_{xt+1} \theta^t \right) \hat{F}_t + q_{xt+1} \theta^t F_t - q_{xt+1} x_t - RP_t$$

We substitute demand of both types of investors into the market clearing condition:

$$\lambda^t \dot{q}_{t+1} + \left(1 - \lambda^t \right) s_{t} = x_t$$

$$\lambda^t \frac{q_{Ft+1} \hat{F}_t + q_{Ft+1} F_t - q_{xt+1} x_t - RP_t}{\alpha V^U_t} + \left(1 - \lambda^t \right) \frac{q_{Ft+1} + q_{xt+1} - q_{xt+1} \theta^t}{\alpha V^U_t} \hat{F}_t + q_{xt+1} \theta^t F_t - q_{xt+1} x_t - RP_t = x_t$$

Matching coefficients with respect to $\hat{F}_t$, $F_t$, and $x_t$ yields the following three equations:

$$\lambda^t \frac{q_{Ft+1} - RP_t \dot{F}_t}{\alpha V^U_t} + \left(1 - \lambda^t \right) \frac{q_{Ft+1} + q_{xt+1} - q_{xt+1} \theta^t - RP_t}{\alpha V^U_t} = 0$$

$$\lambda^t \frac{q_{Ft+1} - RP_t \dot{F}_t}{\alpha V^U_t} + \left(1 - \lambda^t \right) \frac{q_{xt+1} \theta^t - RP_t}{\alpha V^U_t} = 0$$

$$\lambda^t \frac{-q_{xt+1} + RP_t x_t}{\alpha V^U_t} + \left(1 - \lambda^t \right) \frac{-q_{xt+1} + RP_t x_t}{\alpha V^U_t} = 1$$
We first show that \( p_{Ft} \) and \( p_{\hat{F}t} \) sum up to a constant. Sum the first and the second equation:

\[
\lambda_t \frac{q_{Ft+1} + q_{Fx+1} - R_{\hat{F}t} - R_{\hat{F}t}}{\alpha V_t^I} + (1 - \lambda_t) \frac{q_{\hat{F}t+1} + q_{\hat{F}t+1} - R_{Ft} - R_{\hat{F}t}}{\alpha V_t^U} = 0
\]

Factor out \( q_{\hat{F}t+1} + q_{Ft+1} - R_{Ft} - R_{\hat{F}t} \):

\[
\left( q_{\hat{F}t+1} + q_{Ft+1} - R_{Ft} - R_{\hat{F}t} \right) \left[ \lambda_t \frac{1}{\alpha V_t^I} + (1 - \lambda_t) \frac{1}{\alpha V_t^U} \right] = 0
\]

This implies

\[
p_{Ft} + p_{\hat{F}t} = \frac{q_{\hat{F}t+1} + q_{Ft+1}}{R}
\]

\[
= p_{\hat{F}t+1} \left( \frac{1}{R} + \rho^F (1 + p_{Ft+1}) \right)
\]

\[
= p_{\hat{F}t+1} \rho^F + \rho^F (1 + p_{Ft+1})
\]

Solve this equation treating \( p_{Ft} + p_{\hat{F}t} \) as one unknown and impose stationarity

\[
p_{Ft} + p_{\hat{F}t} = \frac{\rho^F}{R - \rho^F}, \forall t
\]

Now focus on the coefficient matching for \( F_t \):

\[
\lambda_t \frac{q_{Ft+1} - R_{Ft}}{\alpha V_t^I} + (1 - \lambda_t) \frac{\rho^F p_{xt+1} \theta_t - R_{Ft}}{\alpha V_t^U} = 0
\]

\[
\lambda_t \frac{q_{\hat{F}t+1} - \rho^F p_{xt+1} \theta_t - R_{\hat{F}t}}{\alpha V_t^I} + (1 - \lambda_t) \frac{\rho^F p_{xt+1} \theta_t - R_{\hat{F}t}}{\alpha V_t^U} = 0
\]

Collect terms related to \( \rho^F p_{xt+1} \theta_t - R_{Ft} \) and move it to the right hand side:

\[
\lambda_t \frac{q_{Ft+1} + p_{xt+1} \theta_t}{\alpha V_t^I} = \left[ \lambda_t \frac{1}{\alpha V_t^I} + (1 - \lambda_t) \frac{1}{\alpha V_t^U} \right] \left( \rho^F p_{xt+1} \theta_t - R_{Ft} \right)
\]

\[
= \left[ \lambda_t \frac{1}{\alpha V_t^I} + (1 - \lambda_t) \frac{1}{\alpha V_t^U} \right] \left( \rho^F p_{xt+1} \frac{p_{Ft}}{p_{xt}} - R \right) p_{Ft}
\]

For the equation of \( x_t \):

\[
\lambda_t \frac{q_{Ft+1} - R_{Ft}}{\alpha V_t^I} + (1 - \lambda_t) \frac{q_{\hat{F}t+1} - R_{\hat{F}t}}{\alpha V_t^U} = 1
\]

\[
\left[ \lambda_t \frac{1}{\alpha V_t^I} + (1 - \lambda_t) \frac{1}{\alpha V_t^U} \right] (-\rho^F p_{xt+1} + R_{xt}) = 1
\]

This is the first equation in Proposition 3.2. Divide the two equations, we obtain the second equation:

\[
\frac{p_{Ft}}{p_{xt}} = \lambda_t \frac{q_{Ft+1} + \rho^F p_{xt+1} \frac{p_{Ft}}{p_{xt}}}{V_t^I}
\]

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Proof of Proposition 3.3

This proposition an extension of Theorem 2 in Grossman and Stiglitz (1980) into an overlapping generation framework. First define expected utility of agents net information cost \( \hat{W}^i \). Plug agents’ budget constraint: \( c_t = (D_{t+1} + P_{t+1} - RP_t)s \) into the utility function, we obtain the expected utility of each type of agent conditional on the realized market price \( P_t \):

\[
\hat{W}^i(P_t) = \max_s EU((D_{t+1} + P_{t+1} - RP_t)s|\Omega^i_t)
\]

Given CARA utility and normally distributed random variables:

\[
\hat{W}^i(P_t) = \max_s EU((D_{t+1} + P_{t+1} - RP_t)s|\Omega^i_t)
\]

\[
= \max_s \left[ -e^{-\alpha(D_{t+1} + P_{t+1} - RP_t)s} \right] |\Omega^i_t|
\]

\[
= \max_s -\exp[-\alpha(E[D_{t+1} + P_{t+1} - RP_t|\Omega^i_t]s - \frac{1}{2} \alpha s^2 Var(D_{t+1} + P_{t+1} - RP_t)]
\]

Hence, maximizing over the objective function is equivalent to maximizing

\[
\max_s E[D_{t+1} + P_{t+1} - RP_t|\Omega^i_t]s - \frac{1}{2} \alpha s^2 Var(D_{t+1} + P_{t+1} - RP_t)
\]

Solve for optimal \( s^* \):

\[
s^* = \frac{E[D_{t+1} + P_{t+1} - RP_t|\Omega^i_t]}{\alpha Var(D_{t+1} + P_{t+1} - RP_t)}
\]

Plug back into the original objective function:

\[
\hat{W}^i(P_t) = -\exp[-\frac{1}{2} \left( \frac{E[D_{t+1} + P_{t+1}|\Omega^i_t] - RP_t)^2}{Var(Q_{t+1}|\Omega^i_t)} \right]
\]

Let

\[
h = Var(Q_{t+1}|\Omega^U_t) - Var(Q_{t+1}|\Omega^i_t) > 0
\]

The reason why \( h > 0 \) is that the information set of the uninformed investors is more coarse than that of the informed investors. Taking the ex-ante conditional expectation of the informed \( \hat{W}^i(P_t) \) with respect to the uninformed’s information set \( \Omega^U_t \):

\[
E[\hat{W}^i(P_t)|\Omega^U_t] = E[-\exp[-\frac{1}{2} \left( \frac{E[D_{t+1} + P_{t+1}|\Omega^i_t] - RP_t)^2}{Var(Q_{t+1}|\Omega^i_t)} \right]
\]

\[
= E[-\exp[-\frac{1}{2} \left( \frac{E[D_{t+1} + P_{t+1}|\Omega^i_t] - RP_t)^2}{h Var(Q_{t+1}|\Omega^i_t)} \right]
\]

\[
= E[-\exp[-\frac{1}{2} \left( \frac{h Var(Q_{t+1}|\Omega^i_t)}{Var(Q_{t+1}|\Omega^i_t)} \right)]
\]

where \( z = \frac{(E[D_{t+1} + P_{t+1}|\Omega^i_t] - RP_t)^2}{\sqrt{h}} \).

Thus, by the moment-generating function of a noncentral chi-squared distribution (formula A.21 of Grossman and Stiglitz (1980)):  

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\[ E[\hat{W}^I(P_t)|\Omega^I] = \frac{1}{\sqrt{1 + \frac{h}{\text{Var}(Q_{t+1}|\Omega_I^I)}}} \exp\left(\frac{-E[z|\Omega^I]^2 \frac{1}{2} h}{1 + \frac{h}{\text{Var}(Q_{t+1}|\Omega_I^I)}}\right) \]

\[ = \sqrt{\frac{\text{Var}(Q_{t+1}|\Omega_I^I)}{\text{Var}(Q_{t+1}|\Omega_I^I)}} W_U(P_t) \]

Integrating on both sides with respect to the current stock price \( P_t \), one gets:

\[ \hat{W}_t^I = \sqrt{\frac{\text{Var}(Q_{t+1}|\Omega_I^I)}{\text{Var}(Q_{t+1}|\Omega_I^I)}} W_t^U \]

Or

\[ \hat{W}_t^I \hat{W}_t^U = \sqrt{\frac{V_t^I}{V_t^U}} \]

Lastly, observe that with information cost \( \hat{W}_t^U = \exp(\alpha R \chi) W_t^U \). Thus

\[ \frac{\hat{W}_t^I}{\hat{W}_t^U} = \exp(-\alpha R \chi) \sqrt{\frac{V_t^I}{V_t^U}} \]

**Proof of Proposition 4.2**

Given equation 4.2, to prove proposition 4.2 it suffices to show that

\[ \lim_{\lambda \to 0} \frac{dp_F}{d\lambda} > 0 \]

To obtain this derivative, one needs to total differentiate the market clearing conditions with respect to \( p_F, p_x, \) and \( \lambda \). This gives us the following result:

\[ \lim_{\lambda \to 0} \frac{dp_F}{d\lambda} = \rho^F \frac{1 + a \left( \frac{\text{Var}(F|\Omega)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right)}{\alpha V^I} p_x > 0 \]

\[ \lim_{\lambda \to 0} \frac{dp_x}{d\lambda} = - \left( 1 + a \left( \frac{\text{Var}(F|\Omega)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \right) \rho^F \text{Var}(F|\Omega) \frac{\rho^F + R}{\alpha} - \rho^F - 2p_x \sigma_x^2 \frac{dp_F}{d\lambda} \]

Please refer to Proposition 3.3 of the previous version of the paper for details available at.

**Proof of Theorem 1**

The value of information is proportional to \( \sqrt{\frac{V_t^I}{V_t^U}} \). To derive its slope, it suffices to check the slope of \( \frac{\Delta V}{V_t^U} \)

\[ \frac{d}{d\lambda} \frac{\Delta V}{V_t^U} = \frac{\frac{d\Delta V}{d\lambda} V_t^I - \frac{dV_t^I}{d\lambda} \Delta V}{(V_t^I)^2} = \left( \frac{d\Delta V}{d\lambda} - \frac{\Delta V}{V_t^U} \frac{dV_t^I}{d\lambda} \right) \]
Thus it suffices to check the sign of
\[
\frac{d\Delta V}{d\lambda} - \frac{\Delta V}{V^J} \frac{dV^J}{d\lambda}
\]
From Proposition 4.2, we know that
\[
\frac{d\Delta V}{d\lambda} = 2 \left( \rho^F + a \phi_F \right) \left( \rho^F - f_F - \rho^x \right) \frac{dp_F}{d\lambda}
\]
For \(\frac{dV^J}{d\lambda}\):
\[
\frac{dV^J}{d\lambda} = \frac{\partial V^J}{\partial p_F} \frac{dp_F}{d\lambda} + \frac{\partial V^J}{\partial p_x} \frac{dp_x}{d\lambda}
\]
We can derive expression for each components when \(\lambda \to 0\) (please refer to the previous version of this paper for derivation details):
\[
\frac{\partial V^J}{\partial p_F} = -2 \left( \rho^F \right)^2 \left( 1 + a \left( \frac{\text{Var}(F|\Omega)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \right) \text{Var}(F|\Omega) \left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \frac{dp_F}{d\lambda}
\]
Substituting in these expressions together with \(\frac{dp_F}{d\lambda}\) and \(\frac{dp_x}{d\lambda}\):
\[
\frac{dV^J}{d\lambda} = -2 \left( \rho^F \right)^2 \left( 1 + a \left( \frac{\text{Var}(F|\Omega)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \right) \text{Var}(F|\Omega) \left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \frac{dp_F}{d\lambda}
\]
Substituting this expression into \(\frac{d\Delta V}{d\lambda} - \frac{\Delta V}{V^J} \frac{dV^J}{d\lambda}\):
\[
\frac{d\Delta V}{d\lambda} - \frac{\Delta V}{V^J} \frac{dV^J}{d\lambda} = 2 \text{Var}(F|\Omega) \left( \rho^F \right)^2 \left( 1 + a \left( \frac{\text{Var}(F|\Omega)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \right) \text{Var}(F|\Omega) \left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \frac{dp_F}{d\lambda}
\]
All other terms are positive except
\[
\left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \rho^F - \rho^x + \frac{\Delta V}{V^J} \left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \rho^F + p_x \sigma_x^2 \frac{\rho^x + R}{\frac{\alpha}{\alpha} - 2p_x \sigma_x^2}
\]
Let
\[
\phi_j = \frac{\Delta V}{V^J} \left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \rho^F + p_x \sigma_x^2 \frac{\rho^x + R}{\frac{\alpha}{\alpha} - 2p_x \sigma_x^2}
\]
where \(p_x^j\) denotes the pricing coefficient associated with high (low) volatility equilibrium. Thus, we have the slope of value of information is upward sloping if and only if:
\[
\left( 1 - \frac{\text{Var}(F|\Omega)}{\sigma_D^2} - \frac{\text{Var}(F|\Omega)}{\sigma_S^2} \right) \rho^F - \rho^x + \phi_j > 0
\]